Optimal Control
Lecture 9

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Suggested ready: Section 3.10 and Section 4.1 of Ref[1] (see class website or the class syllabus for the list of references)

Discrete LQR (Derivation through DP approach)

- For most cases, dynamic programming must be solved numerically - often quite challenging.
- A few cases can be solved analytically - discrete LQR is one of them
- Goal: select control inputs to minimize

$$J = \frac{1}{2} x_N^T H x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k)$$

$$\Rightarrow g_d(x_k, u_k) = \frac{1}{2} x_k^T Q x_k + u_k^T \ldots$$

s.t.

$$x_{k+1} = A_k x_k + B_k u_k$$

- Assume $H = H^T \geq 0$, $Q = Q^T \geq 0$, $R = R^T > 0$
- Including any other constraints greatly complicates the problem.

- Clearly $J_N^*[x_N] = \frac{1}{2} x_N^T H x_N \Rightarrow$ now need to find $J_{N-1}^*[x_{N-1}]$

$$J_{N-1}^*[x_{N-1}] = \min \{ g_d(x_{N-1}, u_{N-1}) + J_N^*[x_N] \}$$

$$= \min_{u_{N-1}} \frac{1}{2} \{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R_N u_{N-1} + x_N^T H x_N \}$$

- Note that $x_N = A_{N-1} x_{N-1} + B_{N-1} u_{N-1}$ so that

$$J_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} \frac{1}{2} [x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1}$$

$$+ (A_{N-1} x_{N-1} + B_{N-1} u_{N-1})^T H (A_{N-1} x_{N-1} + B_{N-1} u_{N-1})]$$

- Take derivative w.r.t. the control inputs

$$\frac{\partial J_{N-1}^*(x_{N-1})}{\partial u_{N-1}} = R_{N-1} u_{N-1} + B_{N-1}^T H (A_{N-1} x_{N-1} + B_{N-1} u_{N-1})$$

Now set this derivative equal to zero

- Recall that both equations are solved backward from k+1 to k

- Now consider time k-1 with

$$J^*_{k-1}(x_{k-1}) = \min_{u_{k-1}} \left\{ \tfrac{1}{2} x^T_{k-1} Q_{k-1} x_{k-1} + u^T_{k-1} R_{k-1} u_{k-1} + J^*_k(x_k) \right\}$$

- Taking derivative with respect to u_{k-1} gives

$$\frac{\partial J^*_{k-1}(x_{k-1})}{\partial u_{k-1}} = u^T_{k-1} R_{k-1} + (A_{k-1} x_{k-1} + B_{k-1} u_{k-1})^T P_k B_{k-1}$$

so that the best control input is

$$u^*_{k-1} = -(R_{k-1} + B^T_{k-1} P_k B_{k-1})^{-1} B^T_{k-1} P_k A_{k-1} x_{k-1}$$

$$= -F_{k-1} x_{k-1}$$

- substitute this control into the expression for $J^*_{k-1}(x_{k-1})$

to show that

$$J^*_{k-1}(x_{k-1}) = \tfrac{1}{2} x^T_{k-1} P_{k-1} x_{k-1}$$

and

$$P_{k-1} = Q_{k-1} + F^T_{k-1} R_{k-1} F_{k-1} + (A_{k-1} - B_{k-1} F_{k-1})^T P_k (A_{k-1} - B_{k-1} F_{k-1})$$

- Thus the same properties hold at time k-1 and k and N and N-1 in particular, so they will always be true.

$$(R_{N-1} + B_{N-1}^T H B_{N-1}) u_{N-1} + B_{N-1}^T H A_{N-1} x_{N-1} = 0$$

which suggests a couple of key things.

- The best control action at time $N-1$ is a linear state feedback on the state at time $N-1$

$$u^*_{N-1} = -(R_{N-1} + B_{N-1}^T H B_{N-1})^{-1} B_{N-1}^T H A_{N-1} x_{N-1}$$

$$\equiv -F_{N-1} x_{N-1}$$

- Furthermore, can show that

$$\frac{\partial^2 J^*_{N-1}(x_{N-1})}{\partial u^2_{N-1}} = R_{N-1} + B_N^T H B_N > 0$$

satisfied because $R_{N-1} > 0$, $H \geq 0$

So that the stationary point is minimum.

- with this control decision, take another look at

$$J^*_{N-1}(x_{N-1}) = \frac{1}{2} x_{N-1}^T \left(Q_{N-1} + F_{N-1}^T R_{N-1} F_{N-1} + (A_{N-1} - B_{N-1} F_{N-1})^T H (A_{N-1} - B_{N-1} F_{N-1}) \right) x_{N-1} \equiv \frac{1}{2} x_{N-1}^T P_{N-1} x_{N-1}$$

- Note that $P_N = H$ which suggests a convenient form for gain F

$$F_{N-1} = [R_{N-1} + B_{N-1}^T P_N B_{N-1}]^{-1} B_{N-1}^T P_N A_{N-1}$$

- Now can continue using <u>induction</u> - assume that at time k the control will be of the form $u_k^* = -F_k x_k$ where

$$F_k = [R_k + B_k^T P_{k+1} B_k]^{-1} B_k^T P_{k+1} A_k$$

and $J_k^*[x_k] = \frac{1}{2} x_k^T P_k x_k$ where

$$P_k = Q_k + F_k^T R_k F_k + (A_k - B_k F_k)^T P_{k+1} (A - B_k F_k)$$

Algorithm.

Can summarize the above in the algorithm.

(i) $P_N = H$

(ii) $F_k = (R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k$

(iii) $P_k = Q_k + F_k^T R_k F_k + (A_k - B_k F_k)^T P_{k+1} (A_k - B_k F_k)$

Cycle through steps ii and iii from $N-1 \to 0$

Note

- the result is a state feedback control with time varying gain, even if A, B, Q, R are constant.
- clear that P_k and F_k are independent of the state and can be computed ahead of time, off-line. (see your earlier notes on LQR)
- Possible to eliminate the F_k part of the cycle and just cycle through P_k

$$P_k = Q_k + A_k^T (P_{k+1} - P_{k+1} B_k (R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1}) A_k$$

Initial assumption $R_k > 0 \ \forall k$ can be relaxed, but we must ensure that

$$(R_{k+1} + B_k^T Q_{k+1} B_k) > 0$$

- In the expression

$$J^*(x_k^i, t_k) = \min_{u_k^{ij}} [g(x_k^i, u_k^{ij}, t_k)\Delta t + J^*(x_{k+1}^j, t_{k+1})]$$

The term $J^*(x_{k+1}^j, t_{k+1})$ plays the role of a "cost to go" which is key concept in DP and other control problems

Calculus of variation and its connection to optimal control

We are going to focus on solving

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}\Big(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t)\Big),\ \ \text{s.t.}$$

$$\dot{x}(t) = f(x(t),u(t),t),$$

$$x(t_0),\ \ t_0 \text{ is given},$$

$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$

$$x(t):\mathbb{R}\rightarrow\mathbb{R}^n,\quad u(t):\mathbb{R}\rightarrow\mathbb{R}^m,\quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\rightarrow\mathbb{R}^n.$$

Observations:

- J is a function of $x(t)$, $u(t)$ both functions over $t\in[t_0,t_f]$
- J is a functional (function of a function)

Static parameter optimization:

- objective: determine a point that minimizes a specific function (the performance measure)

Optimization in continuous-time:

- objective: determine a function that minimizes a specific functional (the performance measure)

Function vs. functional

Def (function): A function f is a rule of correspondence that assigns to each element q in a certain set $\mathcal{D}$ (domain of the function) a unique element in a set $\mathcal{R}$ (range or image of the function)

Def (functional): A functional J is a rule of correspondence that assigns to each function x in a certain class Ω (domain of the functional) a unique real number. The set of real numbers associated with the functions Ω is called the range of the functional.

- functional: function of function
- domain is a class of functions

Example: x: continuous function of t defined in the interval $[t_0, t_f]$ and

$$J(x) = \int_{t_0}^{t_f} x(t)dt.$$

is a functional. Its range is the area under $x(t)$ curves.