Optimal Control
Lecture 7

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Study: Sections 2.2 and (2.4 until the subsection on "An Analytic Solution to the Riccati Equation") of Ref[2]; pay special attention to Theorem 2.4-2 and the discussion following this theorem.

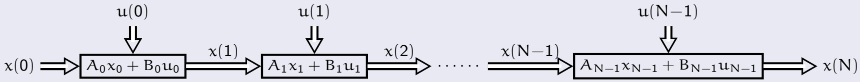
Outline

Optimal control of multi-stage systems over finite horizon

- Finite time optimal optimal LQR (free final state)

Review: Optimal control of multi-stage systems over finite horizon

$$u^\star = \text{argmin} \frac{1}{2} x_N^\top S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \quad \text{s.t.}$$



using 'sweeping method' we can obtain $u_k^\star = -K_k x_k$.

where

$$K_k = (B_k^\top S_{k+1} B_k + R_k)^{-1} B_k^\top S_{k+1} A_k, \quad k = 0, 1, \cdots, N-1.$$

S_k can be calculated off-line from (backward iteration)

$$\begin{cases} S_k = A_k^\top (S_{k+1}^{-1} + B_k R_k^{-1} B_k^\top)^{-1} A_k + Q_k, \; k = N-1, N-2, \cdots, 1, \\ S_N = S_N \quad \text{(given)}. \end{cases}$$

Optimal control gain K_k, even when A, B, R, etc. are time invariant, is time varying!

Review: Optimal LQR over finite horizon

Observations

- optimal control gain K_k, even when A, B, R, etc. are time invariant, is time varying
- time-varying feedback is not always convenient to implement
- need to compute and store sequences of $K_k \in \mathbb{R}^{n \times m}$ control gains.

we may be satisfied by using sub-optimal gain, e.g., a constant gain

Limiting behavior of the Riccati equation

1. When does there exist a bounded S_∞ to the Riccati equation for all choices of S_N?
2. In general, S_∞ depends on S_N. When is S_∞ the same for all choices of S_N?
3. When is the closed-loop plant $A - BK_\infty$ asymptotically stable?

Optimal LQR over finite horizon: steady state solution

Theorem

Let (A, B) be stabilizable. Then, for every choice of S_N, there exists a bounded S_∞ to the Riccati eq. Furthermore, S_∞ is a positive semi-definite solution to ARE

Theorem

Let C be such that $Q = C^\top C \geqslant 0$, and suppose $R > 0$. Supposed (A, C) is observable, then (A, B) is stabilizable if and only if

a) *The is a unique $S_\infty > 0$ to the Riccati equation. Furthermore S_∞ is the unique positive definite solution to ARE.*

b) *The closed-loop plant*

$$x_{k+1} = (A - BK_\infty)x_k$$

is asymptotically stable, where

$$K_\infty = (B^\top S_\infty B + R)^{-1} B^\top S_\infty A.$$

- If plant is observable through the fictitious output, all states are present in J_k. When J_k is small, so are the states
- If (A, C) is unobservable, if the unobservable state goes to infinity it does not effect the cost. Boundedness of cost does not guarantee boundedness of trajectories
- (A, C) detectable is enough
- Choose Q and R wisely. E.g., $Q \in \mathbb{R}^{n\times n}$, $Q = C^\top C > 0 \Rightarrow \text{rank}(C) = n \Rightarrow (A, C)$ observable.

Optimal LQR over finite horizon subject to control input bounds

$$u^\star = \operatorname{argmin} \frac{1}{2} z_N^\top S_N z_N + \frac{1}{2} \sum_{k=0}^{N-1} z_k^\top Q_k z_k + u_k^\top R_k u_k \quad \text{s.t.}$$
$$x(k+1) = Ax(k) + Bu(k)$$
$$z(k) = Cx(k)$$
$$\|u(k)\| \leqslant u_{lim}, \quad k = 0, \cdots, N-1$$

In the following we convert this problem into a more standard optimization problem

$$z(0) = Cx(0)$$
$$z(1) = C\underbrace{(Ax(0) + Bu(0))}_{x(1)} = CAx(0) + CBu(0)$$
$$\vdots$$
$$z(N) = C\underbrace{(Ax(N-1) + Bu(N-1))}_{x(N)} = CA^N x(0) + CA^{N-1}B(0) + \cdots + CBu(N-1)$$

Next, we write these equations in the following

$$\underbrace{\begin{bmatrix} z(0) \\ z(1) \\ \vdots \\ z(N) \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix}}_{G} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ CB & 0 & 0 & & 0 \\ CAB & CB & 0 & & 0 \\ \vdots & & & & \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \cdots & CB \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}}_{U}$$

Optimal LQR over finite horizon subject to control input bounds (con'd)

$$Z = G\,x(0) + H\,U$$

$$\tfrac{1}{2} z_N^\top S_N z_N + \tfrac{1}{2}\sum_{j=0}^{N-1} z_k^\top Q_k z_k = Z^\top \underbrace{\text{Diag}\,(\frac{1}{2}Q_0, \dots, \frac{1}{2}Q_{N-1}, \frac{1}{2}S_N)}_{W_1}\, Z$$

$$\tfrac{1}{2}\sum_{j=0}^{N-1} u_k^\top R_k u_k = U^\top \underbrace{\text{Diag}\,(\frac{1}{2}R_0, \dots, \frac{1}{2}R_{N-1})}_{W_2}\, U$$

$$\begin{aligned} J =& Z^\top W_1\, Z + U^\top W_2\, U = (G\,x(0) + H\,U)^\top W_1 (G\,x(0) + H\,U) + U^\top W_2\, U \\ =& x(0)^\top \underbrace{(G^\top W_1 G)}_{F_3}\, x(0) + (\underbrace{2x(0)^\top G^\top W_1 H}_{F_2})^\top U + \frac{1}{2} U^\top \underbrace{(2(H^\top W_1 H + W2))}_{F_1}\, U \end{aligned}$$

Therefore, our optimal control problem cam be formulated now as

$$U^\star = \text{argmin}\, \frac{1}{2} U^\top F_1 U + F_2^\top U, \quad s.t.$$

$$\begin{bmatrix} I_N \\ -I_N \end{bmatrix} U \leqslant u_{lim}$$

The optimization problem above is in the form of a standard **quadratic program**. There are many standard and efficient codes exists to solve this class of optimization problems (Matlab's QUADPROG is one of those solvers).[1]

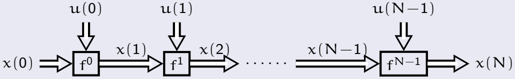
[1] Try to solve problem 4(b) in HW 2 using the formulation here (there is no control constraint in your problem, therefore, you will have an unconstraint optimization problem)

Outline

- Brief introduction on MPC

Optimal control of multi-stage systems over finite horizon- what we did so far

$$u^\star = \text{argmin}\, \underbrace{\phi(x(N)) + \sum_{k=0}^{N-1} L^k(x(k),u(k))}_{J(u(0),\cdots,u(N-1))} \quad \text{s.t.}$$



Our approach so far:

- designed our optimal control $u^\star(k)$ using assumed model and set of constraints (system model)
- nonlinear model has to be cast as static optimization with decision vector of order $O(N)$
- linear system with quadratic cost: for some specific problems we have solution in terms of system matrices

Issues:

- the design is not necessarily closed-loop (especially if you add inequality constraints): modeling error and/or disturbances can deviate the system from the desired output under optimal control.

Use Model Predictive Control (also known as Receding Horizon Control)

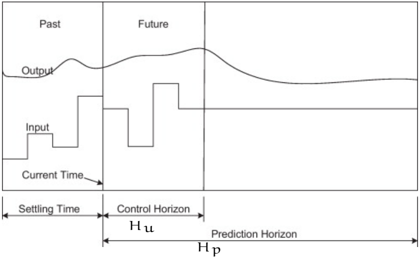
MPC: basic strategy

- At time k, sample the state of the system and use the knowledge of the system model to design an optimal input sequence

$$u(k|k),\ u(k+1|k),\ \cdots,\ u(k+H_u|k)$$

over some finite horizon H_p from the current state $x(k)$. H_u is the control horizon

- Implement a fraction of the input sequence, usually just one step
- repeat for time $k+1$ at state $x(k+1)$.



Usually $H_u << H_p$

- small H_u means fewer variables to compute in the optimization problem at each control interval: faster computations.
- small H_u promotes (but does not guarantee) an internally stable controller.

In our developments below we assume $H_u = H_p$ for simplicity

MPC: some notes

- MPC is a control algorithm that is based on numerically solving on-line an optimization problem subject to equality/inequality constraints at each step
  - can handle systems with nonlinear and time-varying dynamics
  - explicitly accounts for constraints

- the system model can be modified based on the current state of the system

- computationally expensive
  - started in 1970-1980's in process control
  - earlier applications were in slow systems
  - speed of computers increased, now we can use for systems with faster-time scales

Linear MPC (regulation problem)

Given

$$\begin{aligned} \text{system model}: &\ x(k+1) = Ax(k) + Bu(k), \\ \text{measured output}: &\ y(k) = C_y x(k), \\ \text{controlled output}: &\ z(k) = C_z x(k) + D_z u(k), \\ \text{constrained states}: &\ z_c(k) = C_c x(k) + D_c u(k) \end{aligned}$$

subject to

$$\begin{aligned} \Delta u_{\text{min}} \leqslant \Delta u(k) \leqslant \Delta u_{\text{max}}, \\ u_{min} \leqslant u(k) \leqslant u_{\text{max}}, \\ z_{min} \leqslant z_c(k) \leqslant z_{\text{max}}, \end{aligned}$$

find an MPC controller that minimizes

$$J = \sum_{j=0}^{H_p} \|z(k+j|k)\|_Q + \sum_{j=0}^{H_u} \|u(k+j|k)\|_R + \phi(x(k+H_p|k)$$

over H_p prediction horizons.

$\phi(x(k+H_p|k)$ is a terminal cost function (can be used as a tool to induce stability in design)

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How to pick H_p and H_u

- longer horizon have more degree of freedom and take much longer to compute
- if plant model is not very accurate or system is subject to disturbances, planning for longer horizons does not make sense
- smaller H_p can be more suitable for stability

Linear MPC: an example of a typical problem

$$\min_{u} J = \sum_{j=0}^{H_p} (\|z(k+j|k)\|_Q + \sum_{j=0}^{H_p-1} \|u(k+j|k)\|_R$$

$$x(k+j+1|k) = Ax(k+j|k) + Bu(k+j|k),$$

$$x(k|k) = x(k) \text{(use the current state of the system as initial condition)},$$

$$z(k+j|k) = Cx(k+j|k)$$

subject to

$$u_{min} \leqslant u(k) \leqslant u_{\text{max}},$$

Assume $H_p = H_u$

$$------$$

- MPC problem now is cast as quadratic program (QP). Use Matlab quadprog to solve the problem

$$\min_{x} \frac{1}{2} x^\top Hx + f^\top x \text{ such that } \begin{cases} A.x \leqslant b, \\ A_{eq}.x = b_{eq}, \\ x_l \leqslant x \leqslant x_u. \end{cases}$$

- there are also several tool boxed for MPC

MPC: some useful references

- A brief summary on linear MPC and some application examples

  Johan Akesson: "MPCtools 1.0 -Reference Manual". Technical report ISRN LUTFD2/TFRT-7613-SE, Dept. of Automatic Control, Lund Inist. of Tech., Sweden, Jan. 2006. http://www.control.lth.se/media/Education/EngineeringProgram/FRTN15/2012/MPC%20Tools.pdf

  MPCtools software is available to download from:

  http://www.control.lth.se/user/johan.akesson/mpctools/index.html

- Survey Papers
  - C. E. Garcia, D. M. Prett and M. Morari, "Model Predictive Control: Theory and Practice", Automatica, 25(3):335–348, 1989
  - M. Morari, J. H. Lee, "Model predictive control: past, present and future", Computers and Chemical Engineering, 23: 667–682, 1999
  - J.B. Rawlings, "Tutorial: model predictive control technology," American Control Conference, pp. 662-676, 1999
  - D. Q. Mayne, J. B. Rawlings, C. V. Rao, P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality", Automatica 36:789-814, 2000

- Paper on stability mentioned in the class

  A. Bemporad, L. Chisci, E. Mosca: "On the stabilizing property of SIORHC", Automatica, 30(12):2013–2015, 1994.

- Books
  - F. Allgower, A. Zheng, Nonlinear Model Predictive Control, Springer-Verlag, 2000.
  - J. Maciejowski, Predictive Control with Constraints, Pearson Education POD, 2002.
  - Rossiter, J. A., Model-Based Predictive Control: A Practical Approach, CRC Press, 2003