Optimal Control
Lecture 6

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Study: Sections 2.2 and (2.4 until the subsection on "An Analytic Solution to the Riccati Equation") of Ref[2]

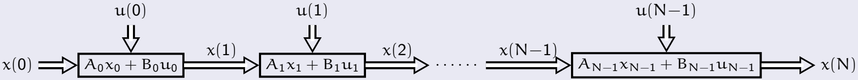
Outline

Optimal control of multi-stage systems over finite horizon

- Finite time optimal optimal LQR (free final state)
- A brief introduction on Model Predictive Control (MPC)

Review: Optimal control of multi-stage systems over finite horizon

$$u^\star = \text{argmin}\,\frac{1}{2}x_N^\top S_N x_N + \frac{1}{2}\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \quad \text{s.t.}$$



$$H^k = \frac{1}{2}x_k^\top Q_k x_k + \frac{1}{2}u_k^\top R_k u_k + \lambda_{k+1}^\top(A_k x_k + B_k u_k), \quad k = 0, 1, \cdots, N-1$$

Free final state: Linear systems with given initial condition

$$\lambda(N) = \frac{\partial \phi(x(N))}{\partial x(N)} \qquad \Longrightarrow \quad \lambda_N = S_N x_N,$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k}, \quad k = 1, \cdots, N-1 \quad \Longrightarrow \quad \lambda_k = Q_k x_k + A_k^\top \lambda_{k+1}, \quad k = 1, \cdots, N-1,$$

$$0 = \frac{\partial H^k}{\partial u_k}, \quad k = 0, \cdots, N-1 \quad \Longrightarrow \quad 0 = R_k u_k + B_k^\top \lambda_{k+1}, \quad k = 0, \cdots, N-1,$$

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}}, \quad k = 1, \cdots, N-1 \quad \Longrightarrow \quad x_{k+1} = A_k x_k + B_k u_k, \quad k = 1, \cdots, N-1,$$

$$x(0) = x_0 \qquad \Longrightarrow \quad x(0) = x_0.$$

ReviewOptimal LQR over finite horizon

using 'sweeping method' we can obtain $u_k^\star = -K_k x_k$.

where

$$K_k = (B_k^\top S_{k+1} B_k + R_k)^{-1} B_k^\top S_{k+1} A_k, \quad k = 0, 1, \cdots, N-1.$$

S_k can be calculated off-line from (backward iteration)

$$\begin{cases} S_k = A_k^\top (S_{k+1}^{-1} + B_k R_k^{-1} B_k^\top)^{-1} A_k + Q_k, \ k = N-1, N-2, \cdots, 1, \\ S_N = S_N \quad (\text{given}). \end{cases}$$

Optimal control gain K_k, even when A, B, R, etc. are time invariant, is time varying!

Observations

- the optimal control gain K_k can be computed off-line and stored
- we can use the current state to generate the input $u = -K_k x_k$
- this is a closed-loop feedback controller

Optimal LQR over finite horizon

- Optimal cost-to-go for $k \in [i, N]$, $i = 0, 1, \cdots, N-1$:

$$J_i^\star = \frac{1}{2} x_i^\top S_i^\star x_i$$

S_k : performance index kernel matrix

- How does cost change for a pre-specified control sequence $\{K_k\}_{k=0}^{N-1}$?

$$J_i = \frac{1}{2} x_N^\top S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k = \frac{1}{2} x_i^\top S_i x_i$$

$$J_i \geqslant J_i^\star$$

Here, for the given set of gains $\{K_k\}_{k=0}^{N-1}$, the corresponding $\{S_k\}_{k=1}^{N-1}$ is generated from

$$S_k = (A_k - B_k K_k)^\top S_{k+1} (A_k - B_k K_k) + K_k^\top R_k K_k + Q_k, \quad k = 1, \cdots, N-1,$$

$$S_N = S_N$$

- Observations
  - optimal control gain K_k, even when A, B, R, etc. are time invariant, is time varying
  - time-varying feedback is not always convenient to implement
  - need to compute and store sequences of $K_k \in \mathbb{R}^{n \times m}$ control gains.

we may be satisfied by using sub-optimal gain, e.g., a constant gain

Optimal LQR over finite horizon: steady state solution

Limiting behavior of the Riccati equation

1. When does there exist a bounded S_∞ to the Riccati equation for all choices of S_N?
2. In general, S_∞ depends on S_N? When is S_∞ the same for all choices of S_N?
3. When is the closed-loop plant $A - BK_\infty$ asymptotically stable?

Theorem

Let (A, B) be stabilizable. Then, for every choice of S_N, there exists a bounded S_∞ to the Riccati eq. Furthermore, S_∞ is a positive semi-definite solution to ARE

Theorem

Let C be such that $Q = C^\top C \geqslant 0$, and suppose $R > 0$. Supposed (A, C) is observable, then (A, B) is stabilizable if and only if

a) *The is a unique $S_\infty > 0$ to the Riccati equation. Furthermore S_∞ is the unique positive definite solution to ARE.*

b) *The closed-loop plant*

$$x_{k+1} = (A - BK_\infty)x_k$$

is asymptotically stable, where

$$K_\infty = (B^\top S_\infty B + R)^{-1} B^\top S_\infty A.$$

Review of stability of discrete-time LTI systems

Theorem

Let $\{\lambda_1, \cdots, \lambda_m\}$, m *be the eignevalues of* $A \in \mathbb{R}^{n \times n}$. *The system* $x(k+1) = Ax(k)$ *is*

- *asymptotically stable if and only of* $|\lambda_i| < 1, \; \forall i = 1, \cdots, m$
- *(marginally stable if* $|\lambda_i| \leqslant 1, \; \forall i = 1, \cdots, m$, *and the eigenvalues with unit modulus have equal algebraic and geometric multiplicity*[a]
- *unstable if* $\exists i$ *such that* $|\lambda_i| > 1$

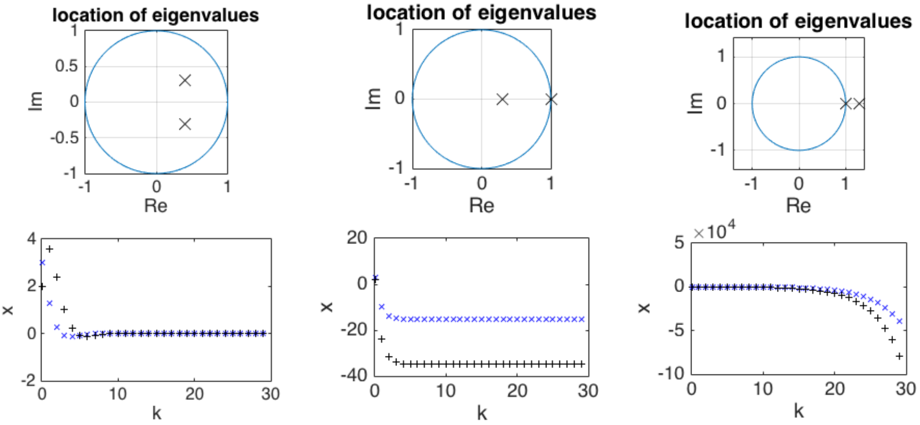
[a] Algebraic multiplicity of λ_i = number of coincident roots λ_i of $\det(\lambda I - A) = 0$. Geometric multiplicity λ_i = number of linearly independent eigenvectors v_i of A corresponding to λ_i.

Review of stability of discrete-time LTI systems: examples

$$x(k+1) = \underbrace{\begin{bmatrix} 0.5 & -0.1 \\ 1 & 0.3 \end{bmatrix}}_{\lambda_{1,2}=0.4\pm 0.3i} x(k), \qquad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x(k+1) = \underbrace{\begin{bmatrix} -5.1 & 2.7 \\ -12.2 & 6.4 \end{bmatrix}}_{\lambda_1=0.3,\ \lambda_2=1} x(k), \qquad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x(k+1) = \underbrace{\begin{bmatrix} -2.1 & 1.7 \\ -6.2 & 4.4 \end{bmatrix}}_{\lambda_1=1.3,\ \lambda_2=1} x(k), \qquad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Review of Lyapunov stability of discrete-time LTI systems

$$x(k+1) = Ax(k), \quad x(0) \in^n \quad (*)$$

Theorem: The following statements are equivalent

- The system (*) is asymptotically stable.
- The system (*) is exponentially stable.
- All the eigenvalues of A have magnitude strictly smaller than 1.
- For every symmetric positive-definite matrix Q, there exists a unique solution P to the following Stein equation (more commonly known as the discrete-time Lyapunov equation)

$$A^\top PA - P = -Q.$$

  Moreover, P is symmetric and positive-definite.
- For every matrix C for which the pair (A, C) is observable, there exists a unique solution P to the Lyapunov equation

$$A^\top PA - P = -C^\top C.$$

  Moreover, P is symmetric, positive-definite.

Optimal LQR over finite horizon: steady state solution

Theorem

Let (A, B) be stabilizable. Then, for every choice of S_N, there exists a bounded S_∞ to the Riccati eq. Furthermore, S_∞ is a positive semi-definite solution to ARE

Theorem

Let C be such that $Q = C^\top C \geqslant 0$, and suppose $R > 0$. Supposed (A, C) is observable, then (A, B) is stabilizable if and only if

a) *The is a unique $S_\infty > 0$ to the Riccati equation. Furthermore S_∞ is the unique positive definite solution to ARE.*

b) *The closed-loop plant*

$$x_{k+1} = (A - BK_\infty)x_k$$

is asymptotically stable, where

$$K_\infty = (B^\top S_\infty B + R)^{-1} B^\top S_\infty A.$$

- If plant is observable through the fictitious output, all states are present in J_k. When J_k is small, so are the states
- If (A, C) is unobservable, if the unobservable state goes to infinity it does not effect the cost. Boundedness of cost does not guarantee boundedness of trajectories
- (A, C) detectable is enough
- Choose Q and R wisely. E.g., $Q \in \mathbb{R}^{n\times n}$, $Q = C^\top C > 0 \Rightarrow \text{rank}(C) = n \Rightarrow (A, C)$ observable.