Optimal Control
Lecture 5

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Reading: Sections 2.1 and 2.2 from Ref.[2]

Optimal control of multi-stage systems over finite horizon

Minimum energy control for linear LTI systems with fix final state

$$u^\star = \text{argmin} \frac{1}{2}\sum_{k=0}^{N-1} u_k^\top R_k u_k, \text{ s.t.}$$
$$x_{k+1} = Ax_k + Bu_k,$$
$$x_0 = x_0, \quad x_N = r_N.$$

$$u_k^\star = R^{-1}B^\top (A^\top)^{N-k-1} G_{0,N}^{-1}(r_N - A^N x_0).$$

where

$$G_{0,N} = \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^\top (A^\top)^{N-i-1}$$

$$= \underbrace{\begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}}_{\mathcal{U}_N} \begin{bmatrix} R^{-1} & & 0 \\ & \ddots & \\ 0 & & R^{-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}^\top$$

- if $|R| \neq 0$, solution exists ($G_{0,N}$ is invertible) if system is reachable ($\mathcal{U}_N$ is full rank)
  - $N \geqslant n$ (recall Cayley-Hamilton theorem)
- This is an open-loop control (depends only on r_N and x_0)
  - If system deviates, there is no way to notice the deviation and respond to it. This is not a robust controller.

Couple of points about this problem.

* if there was no control $x(k+1) = A\,x(k)$, then $x_k = A^k x_0 \Rightarrow x_N = A^N x_0$. Thus $r_N - A^N x_0$ is the difference between the desired and undriven final state; it makes sense that u^*_k should depend on this quantity.

* this is an open-loop control. It can be precomputed knowing only the given x_0 and desired r_N and it is independent of the value of x_k in $k \in [0, N]$. This means that if we apply u^*_k as calculated to the actual system all is well as long as the system model is accurate and nothing occurs to cause x_k to deviate from the optimal trajectory. In practice, however nature is seldom cooperative. Modeling uncertainities and noise cause errors in x_k and the control does not take these errors into account

* to compute $G_{0,N}$ we can use the alternative approach. The solution to the Lyapunov equation

$$P_{k+1} = AP_k + BR^{-1}B^T, \quad k > 0$$

is $P_k = A^k P_0 (A^k)^T + \sum_{i=0}^{k-1} A^{k-i-1} B R^{-1} B^T (A^T)^{k-i-1}$

So if we solve this equation with $P_0 = 0$, then $G_{0,k} = P_k$ for each k. First this recursion is solved to compute $G_{0,N}$ and then U^*_k

Does this control exist?

$G_{0,N}$ should be invertible!

$G_{0,N}$ is the weighted reachability gramian of the system.

In terms of the system reachability matrix $U_K = [B \; AB \; \cdots \; A^{K-1}B]$,

it can be written as

$$G_{0,N} = U_N \begin{bmatrix} R^{-1} & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & R^{-1} \end{bmatrix} U_N^T$$

If $R = I$, then $G_{0,N} = U_N U_N^T$. The optimal control exists iff $|G_{0,N}| \neq$
since we assume $|R| \neq 0$, this is equivalent to U_N having full
rank n, where n is the state dimension ($x \in R^n$). Therefore, we
can drive any x_0 to any desired $x_N = r_N$ for some N iff
the system is reachable. Since reachability implies U_{n+j} has
full rank for all $j \geq 0$, if the system is reachable,
we can guarantee the existence of a control to drive x_0 to
$x_N = r_N$ for any r_N by selecting $N \geq n$.

Bryson (1975)

Sweeping method $\underbrace{\lambda_k = S_k x_k}_{\text{Assume}} \longleftarrow \quad k=0,\dots,N-1$

Lets see if this consistent with the optimality conditions.

$$* \; R_k u_k + B_k^T \lambda_{k+1} = 0 \Rightarrow u_k^* = -R_k^{-1} B_k^T \lambda_{k+1}, \quad k=0,\dots,N-1$$

$$u_k = -R_k^{-1} B_k^T \lambda_{k+1} = -R_k^{-1} B_k^T S_{k+1} x_{k+1}$$

$$x_{k+1} = A_k x_k + B_k(-R_k^{-1} B_k^T S_{k+1}) x_{k+1}$$

$$x_{k+1} = (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k x_k$$

costate eq $\quad \lambda_k = Q_k x_k + A_k^T \lambda_{k+1} \quad k=1,\dots,N-1$

$$S_k x_k = Q_k x_k + A_k^T S_{k+1} x_{k+1}$$

$$S_k x_k = Q_k x_k + A_k^T S_{k+1} (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k x_k$$

has to hold for any x_k

and usually $x_k \neq 0$

$$\boxed{S_k = Q_k + A_k^T S_{k+1} (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k}$$

matrix inversion Lemma $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)VA^{-1}$

$$S_k = A_k^T \left(S_{k+1}^{-1} + B_k R_k^{-1} B_k^T \right)^{-1} A_k + Q_k \longrightarrow \{S_k\}_{k=1}^{N-1}$$

$$S_N$$

$S_N > 0$, $Q_k > 0 \rightarrow S_k > 0$

↳ this can be computed off-line!

$$\lambda_k = S_k x_k \qquad k = 1, \dots, N$$

$$U_k^* = -R_k^{-1} B_k^T \lambda_{k+1} = -R_k^{-1} B_k^T S_{k+1} x_{k+1}$$

$$U_k^* = -\underbrace{(B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k}_{K_k} x_k$$

Kalman Gain $\leftarrow K_k \leftarrow$ can be computed off-line

$$\boxed{U_k^* = -K_k x_k} \rightarrow \text{closed-loop control}$$

↳ Linear Quadratic Regulator (LQR) controller

* Robust to deviation from optimal trajectory (modeled one)

LQR (finite time)

$u_k = -K_k x_k \Rightarrow x_{k+1} = (A_k - B_k K_k) x_k$

@ N: $S_N \leftarrow$ is known

$K_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k$

As good as possible

J^* as small as possible

$S_k = A_k^T S_{k+1} (A_k - B_k K_k) + Q_k$

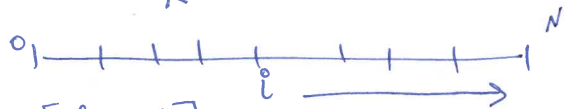
Joseph stabilized form

$S_k = (A - B_k K_k)^T S_{k+1} (A_k - B_k K_k) + K_k^T R K_k + Q_k$

Even if (A, B) time-invarient, optimal gain $\{K_k\}_{k=0}^{N-1}$ is time varying.

* Can I use a suboptimal Control with fixed gain?

$$J = \frac{1}{2} x_N S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^T Q_k x_k + u_k^T R_k u_k$$



Cost to go $[i, N]$

$$J_i = \frac{1}{2} x_N S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} x_k Q_k x_k + u_k^T R_k u_k$$

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observe

$$\frac{1}{2} \sum_{k=i}^{N-1} \left(x_{k+1}^T S_{k+1} x_{k+1} - x_k^T S_k x_k \right) = \frac{1}{2} x_N S_N x_N - \frac{1}{2} x_i^T S_i x_i$$

$$J_i^* = \frac{1}{2} x_i^T S_i x_i \longleftarrow$$

once you have S_i, then you have J_i^*