Optimal Control
Lecture 4

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Reading: Sections 2.1 and 2.2 from Ref.[2]

Outline

Optimal control of multi-stage systems over finite horizon

- first-order conditions for free and constrained final state
- Special case of linear discrete-time systems

Optimal control and its connection to constrained optimization

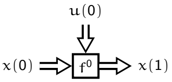
$$u^\star = \underset{u \in \mathbb{R}^m}{\text{argmin}}\ F(x, u), \quad \text{s.t.},$$

$$f(x, u) = 0$$

where $F : \mathbb{R}^{m+n} \to \mathbb{R}$ and $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ are differentiable.

Optimal Control Example

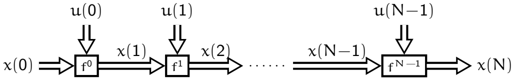
Single stage system



$$u^\star(0) = \text{argmin}\ \underbrace{\phi(x(1)) + L^0(x(0), u(0))}_{J(u(0))}$$

$$\text{s.t.} \quad x(1) = f^0(x(0), u(0)),$$
$$x(0) = x_0 \in \mathbb{R}^n.$$

Multi stage system



$$u^\star = \text{argmin}\ \underbrace{\phi(x(N)) + \sum_{k=0}^{N-1} L^i(x(i), u(i))}_{J(u(0), \cdots, u(N-1))} \quad \text{s.t.}$$

$$x(N) = f^{N-1}(x(N-1), u(N-1)),$$
$$\vdots$$
$$x(1) = f^0(x(0), u(0)),$$
$$x(0) = x_0 \in \mathbb{R}^n.$$

First order optimality condition for single stage optimal control

$$u(0)^\star = \underset{u(0)\in\mathbb{R}^m}{\text{argmin}} \;\; J(x(1), u(0)) = \phi(x(1)) + L^0(x(0), u(0)), \quad \text{s.t.},$$
$$x(1) = f^0(x(0), u(0)), \quad x(1) \in \mathbb{R}^n,\; u(0) \in \mathbb{R}^m,$$
$$x(0) = x_0 \in \mathbb{R}^n, \quad \text{(given initial condition)}.$$

- $\bar{J} = J + \lambda(1)^\top (f^0(x(0), u(0)) - x(1)) = \phi(x(1)) + L^0(x(0), u(0)) + \lambda(1)^\top (f^0(x(0), u(0)) - x(1))$
- Let $H^0(x(0), u(0)), \lambda(1)) = L^0(x(0), u(0)) + \lambda(1)^\top (f^0(x(0), u(0)))$.
- Then, we can rewrite $\bar{J}$ as $\bar{J} = (\phi(x(1)) - \lambda(1)^\top x(1)) + H^0(x(0), u(0), \lambda(1))$.

First order analysis:

$$\bar{J}(x(1) + dx(1), u(0) + du(0)) = \bar{J}(x(1), u(0)) + \underbrace{(\frac{\partial \phi(x(1))}{\partial x(1)} - \lambda(1))^\top dx(1) + (\frac{\partial H^0}{\partial x(0)})^\top dx(0) + (\frac{\partial H^0}{\partial u(0)})^\top du(0)}_{d\bar{J}}$$

Here $dx(0) = 0$ because the initial condition is given (no need for variation). Think of $du(0)$ as free variable and $dx(1)$ the dependent variable, which is defined from the constraint equation (constraint equation relates $dx(1)$ to $du(0)$). Next, pick $\lambda(1)$ such that

$$\frac{\partial \phi(x(1))}{\partial x(1)} - \lambda(1) = 0,$$

which gives us $\bar{J}(x(1) + dx(1), u(0) + du(0)) = \bar{J}(x(1), u(0)) + \underbrace{(\frac{\partial H^0}{\partial u(0)})^\top du(0)}_{d\bar{J}}$.

First order optimality condition for single stage optimal control (cont'd)

- For $(x(1), u(0))$ to be a minimum point we need $d\bar{J} = (\frac{\partial H^0}{\partial u(0)})^\top du(0) \geqslant 0$. Because we are free to vary $du(0)$ in all directions, then the necessary condition for $(x(1), u(0))$ to be a minimum point is $\frac{\partial H^0}{\partial u(0)} = 0$.
- To summarize:
  First order necessary condition for $(x(1), u(0))$ to be a minimum point:

$$\begin{cases} \lambda(1) = \frac{\partial \phi(x(1))}{\partial x(1)}, & n \text{ eq} \\ \frac{\partial H^0}{\partial u(0)} = 0, & m \text{ eq} \\ x(1) = f^0(x(0), u(0)), & n \text{ eq}^1 \end{cases}$$

  Here, we have $2n + m$ equation for $2n + m$ unknowns (the unknowns in the set of equations above are $\lambda(1) \in \mathbb{R}^n$, $x(1) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^m$).

[1] which can also be written as $x(1) = \frac{\partial H^0}{\partial \lambda(1)}$

First order optimality condition for multi-stage optimal control

$$(u^\star(0), \cdots, u^\star(N-1)) = \underset{(u(0)\in\mathbb{R}^m, \cdots, u(N-1)\in\mathbb{R}^m)}{\text{argmin}} \quad J = \phi(x(N)) + \sum_{i=0}^{N-1} L^i(x(i), u(i)), \quad \text{s.t.},$$

$$x(1) = f^0(x(0), u(0)), \quad x(1) \in \mathbb{R}^n,\ u(0) \in \mathbb{R}^m,$$

$$\vdots$$

$$x(N) = f^{N-1}(x(N-1), u(N-1)), \quad x(N) \in \mathbb{R}^n,\ u(N-1) \in \mathbb{R}^m,$$

$$x(0) = x_0 \in \mathbb{R}^n, \quad \text{(given initial condition)}.$$

- $\bar{J} = J + \lambda_1^\top(f^0(x_0, u_0) - x_1) + \cdots + \lambda_N^\top(f^{N-1}(x_{N-1}, u_{N-1}) - x_N) = \phi(x_N) + \sum_{i=0}^{N-1}(L^i(x_i, u_i) + \lambda_{i+1}^\top(f^i(x_i, u_i) - x_{i+1}))$
- Let $H^i(x_i, u_i), \lambda_{i+1}) = L^i(x_i, u_i) + \lambda_{i+1}^\top(f^i(x_i, u_i))$.
- Then, we can rewrite $\bar{J}$ as $\bar{J} = (\phi(x_N) - \lambda_N^\top x_N) + \sum_{i=1}^{N-1}(H^i(x_i, u_i) - \lambda_i^\top x_i) + H^0$.

First order analysis: Here $dx(0) = 0$ because the initial condition is given (no need for variation).

$$d\bar{J} = (\frac{\partial \phi(x_N)}{\partial x_N} - \lambda_N)^\top dx_N + \sum_{i=1}^{N-1}((\frac{\partial H^i}{\partial x_i})^\top dx_i + (\frac{\partial H^i}{\partial u_i})^\top du_i - \lambda_i^\top dx_i) + (\frac{\partial H^0}{\partial x_0})^\top dx_0 + (\frac{\partial H^0}{\partial u_0})^\top du_0$$

$$= (\frac{\partial \phi(x_N)}{\partial x_N} - \lambda_N)^\top dx_N + \sum_{i=1}^{N-1}\Big((\frac{\partial H^i}{\partial x_i} - \lambda_i)^\top dx_i + (\frac{\partial H^i}{\partial u_i})^\top du_i\Big) + (\frac{\partial H^0}{\partial u_0})^\top du_0$$

First order optimality condition for multi-stage optimal control (cont'd)

- Think of $du(i)$, $i = 0, \cdots, N-1$ as free variable and $dx(i+1)$ the dependent variable, which is defined from the constraint equation (constraint equation relates $dx(i+1)$ to $du(i)$).
- Next, pick λ_N such that $$\frac{\partial \phi(x_N)}{\partial x_N} - \lambda_N = 0,$$
- also pick λ_i, such that $$\frac{\partial H^i}{\partial x_i} - \lambda_i = 0, \quad i = 1, \cdots, N-1.$$
- Then, we have
$$d\bar{J} = \sum_{i=1}^{N-1} \Big((\frac{\partial H^i}{\partial u_i})^\top du_i \Big) + (\frac{\partial H^0}{\partial u_0})^\top du_0$$
For $(u(0), \cdots, u(N-1), x(1), \cdots, x(N))$ to be a minimum point we need $d\bar{J} = \sum_{i=1}^{N-1} \Big((\frac{\partial H^i}{\partial u_i})^\top du_i \Big) + (\frac{\partial H^0}{\partial u_0})^\top du_0 \geqslant 0$. Because we are free to vary du_i, $i = 0, \cdots, N-1$ in all directions, then the necessary condition for $(u(0), \cdots, u(N-1), x(1), \cdots, x(N))$ to be a minimum point is $\frac{\partial H^i}{\partial u_i} = 0$, $i = 0, \cdots, N-1$.
- Putting all the conditions we stated and derived, we obtain:
First order necessary condition for $(x(1), u(0))$ to be a minimum point:

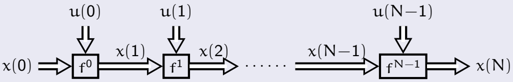
$$\begin{cases} \lambda_N = \frac{\partial \phi(x_N)}{\partial x_N}, & n \text{ eq} \\ \lambda_i = \frac{\partial H^i}{\partial x_i} = 0, \quad i = 1, \cdots, N-1, & (N-1)n \text{ eq} \\ \frac{\partial H^i}{\partial u_i} = 0, \quad i = 0, \cdots, N-1 & N\, m \text{ eq} \\ x_{i+1} = f^i(x_i, u_i), \quad i = 0, \cdots, N-1 & N\, n \text{ eq}^2 \end{cases}$$

Here, we have $2Nn + Nm$ equation for $2Nn + m$ unknowns (the unknowns in the set of equations above are $\lambda_i \in \mathbb{R}^n$, $x_i \in \mathbb{R}^n$ and $u_{i-1} \in \mathbb{R}^m$, $i = 1, \cdots, N$).

[2] which can also be written as $x_{i+1} = \frac{\partial H^i}{\partial \lambda_{i+1}}$

Optimal control of multi-stage systems over finite horizon

$$u^\star = \text{argmin} \underbrace{\phi(x(N)) + \sum_{k=0}^{N-1} L^k(x(k), u(k))}_{J(u(0),\cdots,u(N-1))} \quad \text{s.t.}$$



$$H^k = L^k(x(k), u(k)) + \lambda(k+1)^\top f^k(x(k), u(k)), \quad k = 0, 1, \cdots, N-1$$

Free final state

$$\lambda(N) = \frac{\partial \phi(x(N))}{\partial x(N)},$$

$$\lambda(k) = \frac{\partial H^k}{\partial x(k)}, \ k = 1, \cdots, N-1,$$

$$0 = \frac{\partial H^k}{\partial u(k)}, \ k = 0, \cdots, N-1,$$

$$x(k+1) = \frac{\partial H^k}{\partial \lambda(k+1)}, \ k = 1, \cdots, N-1,$$

$$\begin{cases} x(0) = x_0, & \text{given initial condition,} \\ 0 = \frac{\partial H^0}{\partial x(0)}, & \text{free initial condition.} \end{cases}$$

Constrained final state, i.e., $\psi(x(N)) = 0, \ \psi : \mathbb{R}^n \to \mathbb{R}^p, \ p \leqslant N$

$$\psi(x(N)) = 0,$$

$$\lambda(N) = \frac{\partial(\phi(x(N)) + \sum_{i=1}^p \nu_i \psi_i(x(N)))}{\partial x(N)},$$

$$\lambda(k) = \frac{\partial H^k}{\partial x(k)}, \ k = 1, \cdots, N-1,$$

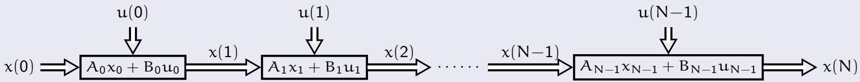
$$0 = \frac{\partial H^k}{\partial u(k)}, \quad k = 0, \cdots, N-1,$$

$$x(k+1) = \frac{\partial H^k}{\partial \lambda(k+1)}, \ k = 1, \cdots, N-1,$$

$$\begin{cases} x(0) = x_0, & \text{given initial condition,} \\ 0 = \frac{\partial H^0}{\partial x(0)}, & \text{free initial condition.} \end{cases}$$

Optimal control of multi-stage systems over finite horizon: regulator problem

$$u^\star = \operatorname{argmin} \frac{1}{2} x_N^\top S_N x_N + \frac{1}{2}\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \quad \text{s.t.}$$



$$H^k = \frac{1}{2} x_k^\top Q_k x_k + \frac{1}{2} u_k^\top R_k u_k + \lambda_{k+1}^\top (A_k x_k + B_k u_k), \quad k = 0, 1, \cdots, N-1$$

Free final state: Linear systems with given initial condition

$$\lambda(N) = \frac{\partial \phi(x(N))}{\partial x(N)} \qquad \Longrightarrow \quad \lambda_N = S_N x_N,$$

$$\lambda(k) = \frac{\partial H^k}{\partial x(k)}, \quad k = 1, \cdots, N-1 \qquad \Longrightarrow \quad \lambda_k = Q_k x_k + A_k^\top \lambda_{k+1}, \quad k = 1, \cdots, N,$$

$$0 = \frac{\partial H^k}{\partial u(k)}, \quad k = 0, \cdots, N-1 \qquad \Longrightarrow \quad 0 = R_k u_k + B_k^\top \lambda_{k+1}, \quad k = 0, \cdots, N-1,$$

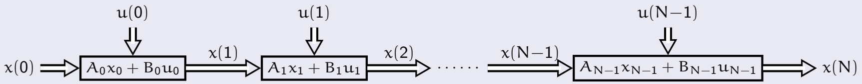
$$x(k+1) = \frac{\partial H^k}{\partial \lambda(k+1)}$$

$$= f^k(x(k), u(k)), \; k = 0, \cdots, N-1 \qquad \Longrightarrow \quad x_{k+1} = A_k x_k + B_k u_k, \quad k = 1, \cdots, N-1,$$

$$x(0) = x_0 \qquad \Longrightarrow \quad x(0) = x_0.$$

Optimal control of multi-stage systems over finite horizon: regulator problem

$$u^\star = \operatorname{argmin} \frac{1}{2} x_N^\top S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \quad \text{s.t.}$$



$$\begin{aligned}
&\lambda_N = S_N x_N, \\
&\lambda_k = Q_k x_k + A_k^\top \lambda_{k+1}, \quad k = 1, \cdots, N-1, \\
&0 = R_k u_k + B_k^\top \lambda_{k+1}, \Rightarrow u^\star = -R_k^{-1} B_k^\top \lambda_{k+1}, \quad k = 0, \cdots, N-1 \\
&x_{k+1} = A_k x_k + B_k u_k, \Rightarrow x_{k+1} = A_k x_k - B_k R_k^{-1} B_k^\top \lambda_{k+1} \quad k = 1, \cdots, N-1, \\
&x(0) = x_0.
\end{aligned}$$

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$$\begin{bmatrix} x(k+1) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} A_k & -B_k R_k^{-1} B_k^\top \\ Q_k & A_k^\top \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k+1) \end{bmatrix}, \quad x(0) = x_0, \ \lambda_N = S_N x_N.$$

If A_k is invertible: $x_k = A_k^{-1} x_{k+1} + A_k^{-1} B_k R_k^{-1} B_k^\top \lambda_{k+1}$. Then, we can write

$$\begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} A_k^{-1} & A_k^{-1} B_k R_k^{-1} B_k^\top \\ Q_k A_k^{-1} & A_k^\top + Q_k A_k^{-1} B_k R_k^{-1} B_k^\top \end{bmatrix} \begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix}, \quad x(0) = x_0, \ \lambda_N = S_N x_N.$$

If we had x_N and λ_N, we could solve the equation above backward in time, but unfortunately we have x_0 and λ_N.

Optimal control of multi-stage systems over finite horizon

Minimum energy control for linear LTI systems with fix final state

$$u^\star = \text{argmin} \frac{1}{2} \sum_{k=0}^{N-1} u_k^\top R_k u_k, \text{ s.t.}$$

$$x_{k+1} = A x_k + B u_k,$$

$$x_0 = x_0, \quad x_N = r_N.$$

$$u_k^\star = R^{-1} B^\top (A^\top)^{N-k-1} G_{0,N}^{-1} (r_N - A^N x_0).$$

where

$$G_{0,N} = \sum_{i=0}^{N-1} A^{N-i-1} B R^{-1} B^\top (A^\top)^{N-i-1}$$

$$= \underbrace{\begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}}_{\mathcal{U}_N} \begin{bmatrix} R^{-1} & & 0 \\ & \ddots & \\ 0 & & R^{-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}^\top$$

- if $|R| \neq 0$, solution exists ($G_{0,N}$ is invertible) if system is reachable ($\mathcal{U}_N$ is full rank)
  - $N \geqslant n$ (recall Cayley-Hamilton theorem)
- This is an open-loop control (depends only on r_N and x_0)
  - If system deviates, there is no way to notice the deviation and respond to it. This is not a robust controller.