Optimal Control
Lecture 12

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Outline

- Calculus of variation
- Optimal Control

Piecewise-smooth extremals

- So far we focused on admissible $x(t)$ that are continuous with continuous first derivatives
- We want to expand to class if piecewise-smooth admissible functions
  - control input is no smooth (e.f., subject to saturation)

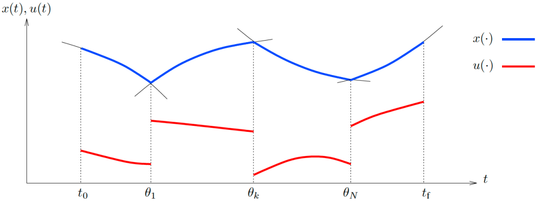

Illustration of a piecewise continuous control $u \in \hat{C}[t_0, t_f]$ (red line), and the corresponding piecewise continuously differentiable response $x \in \hat{C}^1[t_0, t_f]$ (blue line).

  - intermediate state constraints are imposed

Objective: determine vector function $x^\star(t)$ in the class of functions with *piecewise-continuous* first derivative that is a local extremum of

$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}, t) d(t)$$

and respects $x(t_0) = x_0 \in \mathbb{R}^n$, $x(t_f) = x_f$ for given and fixed t_0, x_0, t_f and x_f

Piecewise-smooth extremal

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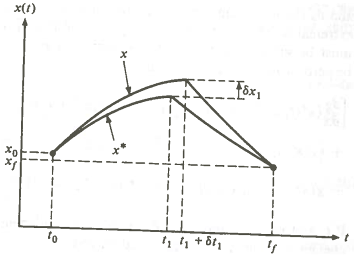
- Assume $\dot{x}$ has a discontinuity at $t_1 \in (t_0, t_f)$, where t_1 is not fixed (or known)

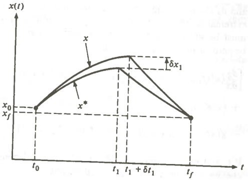
$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}, t) d(t) = \underbrace{\int_{t_0}^{t_1} g(x(t), \dot{x}, t) dt}_{J_1} + \underbrace{\int_{t_1}^{t_f} g(x(t), \dot{x}, t) dt}_{J_2} \text{ as before}$$

$$\begin{aligned}\delta J =& \delta J_1 + \delta J_2 = \\ & \int_{t_0}^{t_1} \left(\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x}\right)dt + g(t_1^-)\delta t_1 + \\ & \int_{t_1}^{t_f} \left(\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x}\right)dt - g(t_1^+)\delta t_1\end{aligned}$$



Piecewise-smooth extremal

$$\delta J = \int_{t_0}^{t_1} (g_x - \frac{d}{dt} g_{\dot{x}}) \delta x dt + g(t_1^-)\delta t_1 + g_{\dot{x}}(t_1^-)\delta x(t_1^-)$$
$$\int_{t_1}^{t_f} (g_x - \frac{d}{dt} g_{\dot{x}}) \delta x dt - g(t_1^+)\delta t_1 - g_{\dot{x}}(t_1^+)\delta x(t_1^+)$$



from lefthand side $\delta x_1 = \delta x(t_1^-) + \dot{x}(t_1^-)\delta t_1,$

from righthand side $\delta x_1 = \delta x(t_1^+) + \dot{x}(t_1^+)\delta t_1,$

- Continuity requires that these two expression for δx_1 be equal
- Already now that it is possible $\dot{x}(t_1^-) \neq \dot{x}(t_1^+)$, so possible that $\delta x(t_1^-) \neq \delta x(t_1^+)$

$$\delta J = \int_{t_0}^{t_1} (g_x - \frac{d}{dt} g_{\dot{x}}) \delta x dt + [g(t_1^-) - g_{\dot{x}}(t_1^-)\dot{x}(t_1^-)]\delta t_1 + g_{\dot{x}}(t_1^-)\delta x_1$$
$$\int_{t_1}^{t_f} (g_x - \frac{d}{dt} g_{\dot{x}}) \delta x dt - [g(t_1^+) - g_{\dot{x}}(t_1^+)\dot{x}(t_1^+)]\delta t_1 - g_{\dot{x}}(t_1^+)\delta x_1$$

$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)] = 0, \quad t \in (t_0, t_f)$$

$$x^\star(t_0) = x_0,$$

Weierstrass-Erdmann Condition $\begin{cases} g_{\dot{x}}(t_1^-) = g_{\dot{x}}(t_1^+), \\ g(t_1^-) - g_{\dot{x}}(t_1^-)\dot{x}(t_1^-) = g(t_1^+) - g_{\dot{x}}(t_1^+)\dot{x}(t_1^+) \end{cases}$

Piecewise-smooth extremal

Typical scenarios that introduce corners is when there exists intermediate constraints

$$x(t_1) = \theta(t_1)$$

- Constraint couples the allowable variations in δx_1 and δt_1

$$\delta x_1 = \frac{d\theta}{dt}\delta t_1 = \dot{\theta}(t_1)\delta t_1$$

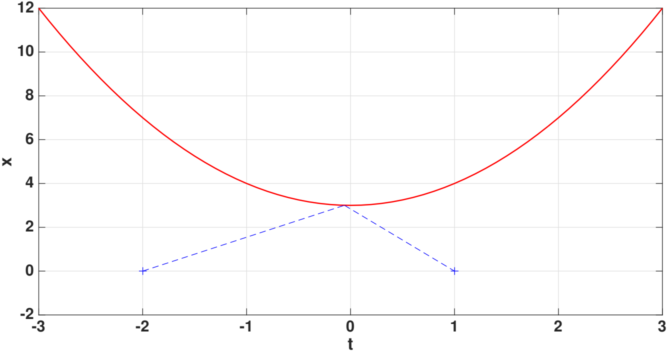
- $\delta J = \int_{t_0}^{t_1}(g_x - \frac{d}{dt}g_{\dot{x}})\delta x dt + [g(t_1^-) - g_{\dot{x}}(t_1^-)\dot{x}(t_1^-)]\delta t_1 + g_{\dot{x}}(t_1^-)\delta x_1 + \int_{t_1}^{t_f}(g_x - \frac{d}{dt}g_{\dot{x}})\delta x dt - [g(t_1^+) - g_{\dot{x}}(t_1^+)\dot{x}(t_1^+)]\delta t_1 - g_{\dot{x}}(t_1^+)\delta x_1$
  $\delta J = \int_{t_0}^{t_1}(g_x - \frac{d}{dt}g_{\dot{x}})\delta x dt + [g(t_1^-) - g_{\dot{x}}(t_1^-)\dot{x}(t_1^-)]\delta t_1 + g_{\dot{x}}(t_1^-)(\dot{\theta}(t_1^-)\delta t_1) + \int_{t_1}^{t_f}(g_x - \frac{d}{dt}g_{\dot{x}})\delta x dt - [g(t_1^+) - g_{\dot{x}}(t_1^+)\dot{x}(t_1^+)]\delta t_1 - g_{\dot{x}}(t_1^+)(\dot{\theta}(t_1^+)\delta t_1)$

$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)] = 0, \quad t \in (t_0, t_f)$$
$$x^\star(t_0) = x_0,$$
$$g(t_1^-) + g_{\dot{x}}(t_1^-)[\dot{\theta}(t_1^-) - \dot{x}(t_1^-)] = g(t_1^+) + g_{\dot{x}}(t_1^+)[\dot{\theta}(t_1^+) - \dot{x}(t_1^+)]$$
$$x(t_1) = \theta(t_1)$$

- Note here that $g_{\dot{x}}(t_1^-) = g_{\dot{x}}(t_1^+)$ no longer is needed. Instead we have $x(t_1) = \theta(t_1)$.

Piecewise-smooth extremal: Example

Example: Find the shortest length path joining the points $x = 0$, $t = -2$ and $x = 0$ and $t = 1$ that touches the curve $x = t^2 + 3$ at some point.



Solution is the dashed blue lines

Optimal control

We are going to focus on solving

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t))dt, \quad \text{s.t.}$$

$$\dot{x}(t) = a(x(t),u(t),t),$$

$$x(t_0),\ t_0 \text{ is given},$$

$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$

$$x(t): \mathbb{R} \to \mathbb{R}^n, \quad u(t): \mathbb{R} \to \mathbb{R}^m, \quad f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n.$$

- Use Lagrange multiplier to write

$$J = h(x(t_f),t_f) + \int_{t_0}^{t_f} \big(g(x(t),u(t),t) + p(t)^\top (a(x(t),u(t),t) - \dot{x}(t))\big)dt$$

Optimal control

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$$\dot{x}(t) = a(x(t),u(t),t),$$

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$$x(t):\mathbb{R}\to\mathbb{R}^n, \quad u(t):\mathbb{R}\to\mathbb{R}^m, \quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n.$$

- Use Lagrange multiplier to write

$$J = h(x(t_f),t_f) + \int_{t_0}^{t_f}\big(g(x(t),u(t),t) + p(t)^\top(a(x(t),u(t),t) - \dot{x}(t))\big)dt$$

- Define the Hamiltonian to help with sorting out the equations

$$\boxed{H(x,u,p,t) = g(x(t),u(t),t) + p(t)^\top a(x(t),u(t),t)}$$

Optimal control

- $$\delta J_a = (h_x - p(t_f))^\top \delta x_f + \left[h_{t_f} + g + p^\top(a - \dot{x}) + p^\top \dot{x}\right]_{t_f} \delta t_f + $$
$$+ \int_{t_0}^{t_f} \left[(H_x + \dot{p})^\top \delta x(t) + H_u^\top \delta u(t) + (a - \dot{x})^\top \delta p(t)\right] dt$$

first order conditions for extremal solution

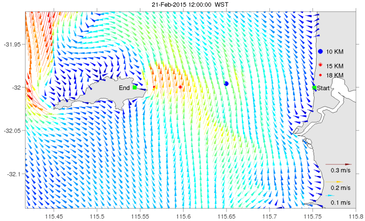
$$\dot{p} = -H_x, \qquad (n \text{ dimensional})$$
$$0 = H_u, \qquad (m \text{ dimensional})$$
$$0 = H_p \rightarrow \quad \dot{x} = a(x, u, t), \qquad (n \text{ dimensional})$$

Boundary conditions $x(t_0) = x_0$, and

- if t_f free
$$h_{t_f} + g + p^\top a = h_{t_f} + H(t_f) = 0$$
- if $x_i(t_f)$ is fixed: $x_i(t_f) = x_{i_f}$
- if $x_i(t_f)$ is free, then $p_i(t_f) = \frac{\partial h}{\partial x_i}(t_f)$

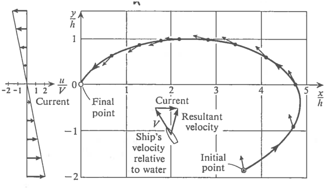
Constrained functional optimization: example

- Minimum-time path through a region of position dependent vector velocity (Zernelo's problem)



Example of an ocean current vector field

  - The forward velocity of the ship V is constant but its steering angle θ can be controlled.
  - in the depicted example, it is assume that the current's velocity vector is only in x direction



see [Bryson and Ho]

Next couple of slides are for self-study

Constrained functional optimization

Determine vector function $w^\star(t) : \mathbb{R} \to \mathbb{R}^{n+m}$ in the class of functions with continuous first derivative that is a local extremum of

$$J(w(t), t) = \int_{t_0}^{t_f} g(w(t), \dot{w}, t) d(t)$$

and respects

$$\begin{aligned} &f(w(t), \dot{w}(t), t) = 0_n, \text{ set of n differential equations,} \\ &w(t_0) = w_0 \in \mathbb{R}^{n+m}, \\ &w(t_f) = w_f \quad \text{various terminal conditions possible.} \end{aligned}$$

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- in control problems $w = \begin{bmatrix} u \in \mathbb{R}^m \\ x \in \mathbb{R}^n \end{bmatrix}$

To derive the first order necessary conditions we proceed with the following

- Similar to function parameter optimization, augment the cost functional with the constraint using Lagrange multiplier

$$J_a(w(t), t) = \int_{t_0}^{t_f} \Big(g(w(t), \dot{w}, t) + p(t)^\top f(w(t), \dot{w}, t) \Big) d(t)$$

- Notice that $p(t)$ is is time-varying (gives more degree of freedom)
- If constraint is satisfied the augmented cost functional is same as $J(w(t), t)$

Constrained functional optimization (cont'd)

- Let
$$g_a(w(t), p(t), \dot{w}(t), t) \equiv g(w(t), \dot{w}(t), t) + p(t)^\top f(w(t), \dot{w}(t), t)$$
- Then
$$J_a(w(t), p(t), \delta x, \delta p, t) = \int_{t_0}^{t_f} \Big(g_a(w(t), p(t), \dot{w}, t) \Big) d(t)$$
- Invoke Fundamental Theorem of Calculus of Variation: $\delta J_a(w^\star(t), p^\star(t), \delta x, \delta p, t) = 0$
- The variations δx and δp are independent from one another.

The first order necessary conditions are

$$\text{Euler equation :} \begin{cases} \frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial w} - \frac{d}{dt}\left[\frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial \dot{w}}\right] = 0, \\ \frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial p} - \frac{d}{dt}\left[\frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial \dot{p}}\right] = 0, \end{cases}$$

$$\begin{cases} \frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial w} - \frac{d}{dt}\left[\frac{\partial g_a(w(t),p(t),\dot{w},t)}{\partial \dot{w}}\right] = 0, & (n+m \text{ dimensional}) \\ w(t_0) = w_0, \\ w(t_f) = w_f, \end{cases}$$

$$f(w, \dot{w}, t) = 0, \qquad (n \text{ dimensional}).$$

- $w^\star(t) : \mathbb{R} \to \mathbb{R}^{n+m}$, $p^\star(t) : \mathbb{R} \to \mathbb{R}^n$