Optimal Control
Lecture 11

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Suggested ready: Section 4.1 and 4.2 of Ref[1] (see class website or the class syllabus for the list of references)

Review: optimal control problems of interest

$$x^\star(t)\Big|_{t\in[t_0,t_f]} = \text{argmin}\left(J(x(t)) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt\right) \text{ s.t.}$$

$$x(t_0) = x_0,$$

$$x(t_f) = x_f \quad \text{(various terminal conditions)}$$

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t)dt), \quad \text{s.t.}$$

$$\dot{x}(t) = f(x(t),u(t),t),$$

$x(t_0)$, t_0 is given,

$m(x(t_f),t_f) = 0 \leftarrow$ when final state is constrained,

$$x(t) : \mathbb{R} \to \mathbb{R}^n, \quad u(t) : \mathbb{R} \to \mathbb{R}^m, \quad f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n.$$

Review: extremal of a functional: fundamental theorem of the calculus of variation

Minimizer of a function f(q) is $q^\star$ if

$$f(q^\star) \leqslant f(q)$$

for all admissible q in $\|q - q^\star\| \leqslant \epsilon$

Minimizer of a functional $J(x(t))$ is $x^\star(t)$ if

$$J(x^\star(t)) \leqslant J(x(t))$$

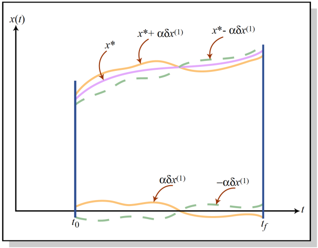
for all admissible $x(t)$ in $\|x(t) - x^\star(t)\| \leqslant \epsilon$.

Fundamental theorem of the calculus of variation

- Let x be a vector function of t in the class Ω, and $J(x)$ be a differential functional of x.
- Assume that all $x \in \Omega$ are not constrained by any boundaries. If $x^\star$ is an extremal function, the variation of J must vanish in $x^\star$

$$\delta J(x^\star, \delta x) = 0$$

for all admissible $x \in \Omega$.



First order necessary optimality conditions

$$x^\star(t)\Big|_{t\in[t_0,t_f]} = \text{argmin}\left(J(x(t)) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt\right) \text{ s.t.}$$

$$x(t_0) = x_0,$$

$$x(t_f) = x_f \quad \text{(various terminal conditions)}$$

Variation

$$\delta J(x(t),\delta x(t)) = g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f + g_{\dot{x}}(x(t_f),\dot{x}(t_f),t_f)\,\delta x(t_f) + \int_{t_0}^{t_f}\left(g_x(x(t),\dot{x}(t),t) - \frac{d}{dt}g_{\dot{x}}(x(t),\dot{x}(t),t)\right)\delta x(t)\,dt$$

- Both t_f and $x(t_f)$ are specified and are given
  - In this case $\delta t_f = 0$ and $\delta x(t_f) = 0$
  - $\delta J(x(t),\delta x(t)) = \int_{t_0}^{t_f}\left(g_x(x(t),\dot{x}(t),t) - \frac{d}{dt}g_{\dot{x}}(x(t),\dot{x}(t),t)\right)\delta x(t)\,dt = 0 \Rightarrow$

the (first order) necessary condition for a maximum or minimum

$$g_x(x(t),\dot{x}(t),t) - \frac{d}{dt}g_{\dot{x}}(x(t),\dot{x}(t),t) = 0 \qquad \text{Euler Equation}$$

$$x(0) = x_0,$$

$$x(t_f) = x_f.$$

Final time is specified but $x(t_f)$ is free

Variation

$$\delta J(x(t), \delta x(t)) = g(x(t_f), \dot{x}(t_f), t_f)\, \delta t_f + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\, \delta x(t_f) + \int_{t_0}^{t_f} \left(g_x(x(t), \dot{x}(t), t) - \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t) \right) \delta x(t)\, dt$$

- Final time t_f specified, but $x(t_f)$ is free
  - In this case $\delta t_f = 0$, but$\delta x(t_f) \neq 0$
  - $\delta J(x(t), \delta x(t)) = g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\, \delta x(t_f) + \int_{t_0}^{t_f} \left(g_x(x(t), \dot{x}(t), t) - \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t) \right) \delta x(t)\, dt = 0 \Rightarrow$

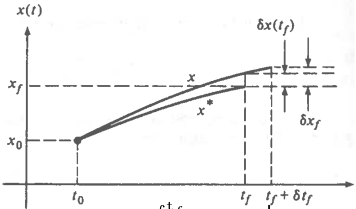
the (first order) necessary condition for a maximum or minimum

$$g_x(x(t), \dot{x}(t), t) - \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t) = 0$$

$$g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) = 0, \quad t_f \text{ is known},$$

$$x(0) = x_0$$

Free terminal time: both final time t_f and $x(t_f)$ are free



$$\delta x(t_f) \neq 0, \quad \delta t_f \neq 0$$
$$\delta x_f \neq \delta x(t_f),$$
$$\delta x_f \approx \delta x(t_f) + \dot{x}^\star(t_f)\delta t_f$$
$$\Rightarrow \delta x(t_f) = \delta x_f - \dot{x}^\star(t_f)\delta t_f,$$

substitute in $\delta J(x(t), \delta x)$ to obtain:

$$\delta J(x(t), \delta x) = \int_{t_0}^{t_f} \{(g_x - \frac{d}{dt} g_{\dot{x}}) \cdot \delta x(t)\}dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x(t_f) + g(x(t_f), \dot{x}(t_f), t_f)\delta t_f =$$

$$\int_{t_0}^{t_f} (g_x - \frac{dg_{\dot{x}}}{dt}) \cdot \delta x(t) dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x_f + (g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}^\star(t_f))\delta t_f = 0$$

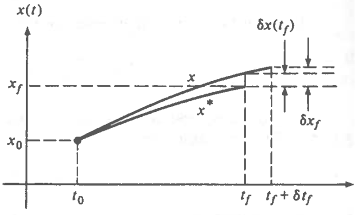
Any extremum $x^\star(t)$ should satisfy

$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)] = 0,$$
$$x^\star(t_0) = x_0,$$

depending on the relationship between $x(t_f)$ and t_f, different set of terminal boundary conditions are obtained

1. Unrelated
2. related by $x(t_f) = \Theta(t)$
3. constrained relationship $m(x(t_f), t_f) = 0$

Free terminal time: both final time t_f and $x(t_f)$ are free and unrelated



$$\delta x(t_f) \neq 0, \quad \delta t_f \neq 0$$
$$\delta x_f \neq \delta x(t_f),$$
$$\delta x_f \approx \delta x(t_f) + \dot{x}^\star(t_f)\delta t_f$$

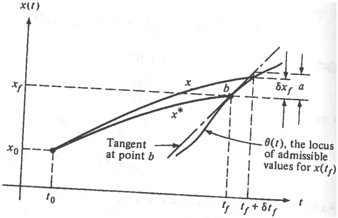
$$\delta J(x(t), \delta x) = \int_{t_0}^{t_f} \left(g_x - \frac{dg_{\dot{x}}}{dt}\right) \cdot \delta x(t) dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \delta x_f + \left(g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}^\star(t_f)\right)\delta t_f = 0$$

Any extremum $x^\star(t)$ should satisfy

1- t_f and $x(t_f)$ are free and unrelated $\Rightarrow$ δt_f and δx_f are independent and arbitrary

$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)\right] = 0,$$
$$x^\star(t_0) = x_0,$$
$$g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) = 0,$$
$$g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}^\star(t_f) = 0.$$

Free terminal time: both final time t_f and $x(t_f)$ are free but related through $x(t_f) = \Theta(t_f)$



final time and final state are free, but related

$$\delta x(t_f) \neq 0, \quad \delta t_f \neq 0$$
$$\delta x_f \neq \delta x(t_f),$$
$$\delta x_f \approx \delta x(t_f) + \dot{x}^\star(t_f)\delta t_f$$

$$x(t_f) = \Theta(t_f) \Rightarrow \delta x_f = \left.\frac{d\Theta}{dt}\right|_{t_f} \delta t_f$$

$$\delta J(x(t), \delta x) = \int_{t_0}^{t_f} (g_x - \frac{dg_{\dot{x}}}{dt}) \cdot \delta x(t) dt + \Big(g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \left.\frac{d\Theta}{dt}\right|_{t_f} + g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}^\star(t_f) \Big) \delta t_f = 0$$

Any extremum $x^\star(t)$ should satisfy

2- t_f and $x(t_f)$ are free and but related through $x(t_f) = \Theta(t_f)$

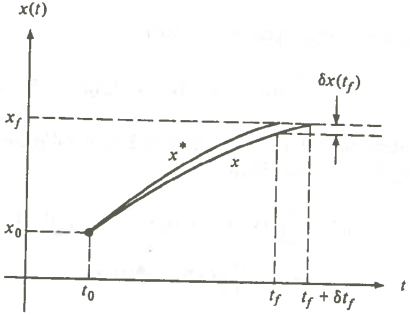
$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)] = 0,$$

$$x^\star(t_0) = x_0,$$

$$x(t_f) = \Theta(t_f),$$

$$g(x(t_f), \dot{x}(t_f), t_f) + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot (\left.\frac{d\Theta}{dt}\right|_{t_f} - \dot{x}^\star(t_f)) = 0. \quad \text{(Transversality condition)}$$

Free final time but fixed and pre-specified final state



t_f is unknown.

$\delta x(t_f)$ is neither zero or free, it depends on $\delta t_f \neq 0$

$$\delta x(t_f) + \dot{x}(t_f)\delta t_f = 0 \quad \text{(see the fig)}$$

$$\delta x(t_f) = -\dot{x}(t_f)\delta t_f, \quad \delta t_f \neq 0$$

$$\delta J(x(t), \delta x) = \int_{t_0}^{t_f} \{(g_x - \frac{d}{dt} g_{\dot{x}}) \cdot \delta x(t)\} dt + \left(g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f) \cdot \dot{x}(t_f)\right)\delta t_f = 0$$

The first order necessary conditions are

$$\frac{\partial g}{\partial x}(x^\star(t), \dot{x}^\star(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^\star(t), \dot{x}^\star(t), t)] = 0,$$
$$x^\star(t_0) = x_0,$$
$$x^\star(t_f) = x_f,$$
$$g(x^\star(t_f), \dot{x}^\star(t_f), t_f) - g_{\dot{x}}(x^\star(t_f), \dot{x}^\star(t_f), t_f) \cdot \dot{x}^\star(t_f) = 0.$$

Constrained terminal states

Determine vector function $x^\star(t)$ in the class of functions with continuous first derivative that is a local extremum of

$$J(x(t), t) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}, t) d(t)$$

and respects

$$x(t_0) = x_0,$$
$$m(x(t_f), t_f) = 0, \quad t_f \text{ can be free.}$$

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- use Lagrange multiplier ν to obtain the augmented cost functional

$$J_a(x(t), t) = h(x(t_f), t_f) + \nu^\top m(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), \dot{x}, t) dt$$

  - when constraint is satisfied J and J_a are the same
- Invoke Fundamental Theorem of Calculus of Variation: $\delta J_a = 0$
  - The variations are in δx, $\delta \nu$, $\delta x(t_f)$, and δt_f (they are not all independent from one another).

Constrained terminal states

- The variations are in δx, $\delta \dot{x}$, $\delta \nu$, $\delta x(t_f)$, and δt_f

$$\delta J_a = h_x(t_f)\delta x_f + h_{t_f}\delta t_f + m(t_f)^\top \delta \nu + \nu^\top (m_x(t_f)\delta x_f + m_{t_f}(t_f)\delta t_f) + \int_{t_0}^{t_f} [g_x \delta x + g_{\dot{x}} \delta \dot{x}] dt + g(t_f)\delta t_f$$

- The variations are not all independent from one another

$$\delta \dot{x} = \frac{d}{dt}\delta x,$$
$$\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f,$$

$$\delta J_a = [h_x(t_f) + \nu^\top m_x(t_f) + g_{\dot{x}}]\delta x_f + [h_{t_f} + \nu^\top m_{t_f}(t_f) + g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f)]\delta t_f + m^\top(t_f)\delta \nu + \int_{t_0}^{t_f} [g_x \delta x - \frac{d}{dt} g_{\dot{x}}]\delta x dt$$

- Let $w(x(t_f), \nu, t_f) = h(x(t_f), t_f) + \nu^\top m(x(t_f), t_f)$
-

$$\delta J_a = [w_x(t_f) + g_{\dot{x}}]\delta x_f + [w_{t_f} + g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f)]\delta t_f + m^\top(t_f)\delta \nu + \int_{t_0}^{t_f} [g_x \delta x - \frac{d}{dt} g_{\dot{x}}]\delta x dt$$

Constrained terminal states

$$\delta J_a = [w_x(t_f) + g_{\dot{x}}]\delta x_f + [w_{t_f} + g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f)]\delta t_f + m^\top(t_f)\delta \nu + \int_{t_0}^{t_f} [g_x \delta x - \frac{d}{dt} g_{\dot{x}}]\delta x dt$$

first order conditions for extremal solution

$$\frac{\partial g(x(t),\dot{x},t)}{\partial x} - \frac{d}{dt}[\frac{\partial g(x(t),\dot{x},t)}{\partial \dot{x}}] = 0, \qquad (n \text{ dimensional})$$

$$x(t_0) = x_0, \qquad (n \text{ dimensional})$$

$$m(x(t_f), t_f) = 0, \qquad (m \text{ dimensional})$$

$$w_x(t_f) + g_{\dot{x}} = 0, \qquad (n \text{ dimensional})$$

$$w_{t_f} + g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f) = 0, \qquad (1 \text{ dimensional})$$

- t_f is fixed we lose the last condition in the box above
- t_f is fixed, $x(t_f)$ is free, then there is no m and no need for ν and $w = h$
- see Kirk (Ref [1]) book for various other conditions