Optimal Control
Lecture 10

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Suggested ready: Section 4.1 and 4.2 of Ref[1] (see class website or the class syllabus for the list of references)

Calculus of variation and its connection to optimal control

We are going to focus on solving

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}\left(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t)\right), \quad \text{s.t.}$$

$$\dot{x}(t) = f(x(t),u(t),t),$$

$$x(t_0),\ t_0 \text{ is given},$$

$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$

$$x(t):\mathbb{R}\rightarrow\mathbb{R}^n, \quad u(t):\mathbb{R}\rightarrow\mathbb{R}^m, \quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\rightarrow\mathbb{R}^n.$$

Observations:

- J is a function of $x(t)$, $u(t)$ both functions over $t\in[t_0,t_f]$
- J is a functional (function of a function)

Static parameter optimization:

- objective: determine a point that minimizes a specific function (the performance measure)

Optimization in continuous-time:

- objective: determine a function that minimizes a specific functional (the performance measure)

Function vs. functional

Def (function): A function f is a rule of correspondence that assigns to each element q in a certain set $\mathcal{D}$ (domain of the function) a unique element in a set $\mathcal{R}$ (range or image of the function)

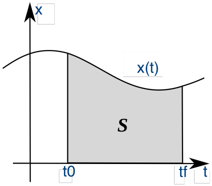
Def (functional): A functional J is a rule of correspondence that assigns to each function x in a certain class Ω (domain of the functional) a unique real number. The set of real numbers associated with the functions Ω is called the range of the functional.

- functional: function of function
- domain is a class of functions

Example: x: continuous function of t defined in the interval $[t_0, t_f]$ and

$$J(x) = \int_{t_0}^{t_f} x(t)dt.$$

is a functional. Its range is the area under $x(t)$ curves.



Calculus of variation

- discrete-time optimal control: can be cast as parameter optimization with a finite dimensional decision variable and constraints

- continuous-time optimal control: infinite dimensional decision variable
  - Continuous-time optimal control: use Calculus of Variation

Calculus of Variation

- field of mathematical analysis that deals with maximization/minimization of functionals
- functionals are defined as integrals involving functions and their derivatives
- interest is in extremal functions that make the functional attain
  - maximum
  - minimum
  - or stationary functions (those where the rate of change of the functional is zero)

Extremal of a functional: fundamental theorem of the calculus of variation

$$q^\star = \text{argmin}\, f(q)$$

Point $q^\star$ is a minimizer of a function $f(q)$ iff

$$f(q^\star) \leqslant f(q)$$

for all admissible q in $\|q - q^\star\| \leqslant \epsilon$

$$x^\star = \text{argmin}\, J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)$$

Function $x^\star(t)$ is a minimizer of functional $J(x(t))$ iff

$$J(x^\star(t)) \leqslant J(x(t))$$

for all admissible $x(t)$ in $\|x(t) - x^\star(t)\| \leqslant \epsilon$.

Tools we need for our studies:

- How to measure closeness of two functions?
- How to compute/approximate variation of a functional due to ‘small’ changes in its arguments, which are functions?

closeness of functions

Norm:

in n-dimensional Euclidean space: rule of correspondence which assigns to each point q a real number.

1. $\|q\| \geqslant 0$ and $\|q\| = 0$ iff $q = 0$
2. $\|\alpha q\| = |\alpha| \|q\|$ for all $\alpha \in \mathbb{R}$
3. $\|q^1 + q^2\| \leqslant \|q^1\| + \|q^2\|$

q^1 and q^2 close together $\Leftrightarrow$ $\|q^1 - q^2\|$ is small

of a function: rule of correspondence which assigns to each function $x \in \Omega$, defined for $t \in [t_0, t_f]$, a real number.

1. $\|x\| \geqslant 0$ and $\|x\| = 0$ iff $x(t) = 0$ for all $t \in [t_0, t_f]$
2. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{R}$
3. $\|x^1 + x^2\| \leqslant \|x^1\| + \|x^2\|$

Intuitively speaking norm of the difference of two functions should be

- zero if the functions are identical
- small, if the functions are "close"
- large if the functions are "far apart"

Examples

- $\|x\|_2 = (\int_{t_0}^{t_f} x^\top(t) x(t) dt)^{1/2}$
- $\|x\| = \max_{t_0 \leqslant t \leqslant t_f} (|x(t)|)$, (scalar x)

Increment of functional

Increment:

of a function f: If q, $q + \Delta q \in \mathcal{D}$, the increment of f is

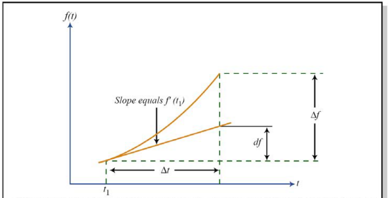
$$\Delta f = f(q + \Delta q) - f(q).$$

of a functional J: If x and $x + \delta x$ are functions for which the functional J is defined, then increment of J is

$$\Delta J = J(x + \delta x) - J(x).$$

δx is the *variation* of the function x

The variation of a functional



Differential *df* versus increment *Δf* shown for a function, but the same difference holds for a functional.

variation of a functional ∼ differential of a function

The increment of a functional can be written as

$$\Delta J(x(t), \delta x(t)) = \delta J(x(t), \delta x(t)) + g(x(t), \delta x(t)).\|\delta x(t)\|,$$

where δJ is a linear in $\delta x(t)$. If

$$\lim_{\|\delta x(t)\| \to 0} (g(x(t), \delta x(t))) = 0,$$

then J is said to be *differentiable* on x and δJ is the variation of J evaluated for a function x.

A *variation of* the functional is a linear approximation of this increment, i.e., $\delta J(x(t), \delta x(t))$ is linear in $\delta x(t)$.

$$\Delta J(x(t), \delta x(t)) = \delta J(x(t), \delta x(t)) + \text{H.O.T.},$$

The variation of a functional: example 1

How to compute variation of $J(x(t)) = \int_{t_0}^{t_f} f(x(t))dt$ (assuming f has first and second continuous derivative)?

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} \frac{\partial f(x(t))}{\partial x(t)} \cdot \delta x dt + f(x(t_f))\delta t_f - f(x(t_0))\delta t_0$$

See next page for the derivation

The variation of a functional: example 1 (cont'd)

$$\begin{aligned}
\Delta J(x(t), \delta x(t)) &= J(x(t) + \delta x(t)) - J(x(t)) \\
&= \int_{t_0+\delta t_0}^{t_f+\delta t_f} (f(x(t) + \delta x(t))\, dt - \int_{t_0}^{t_f} f(x(t))\, dt \\
&= -\int_{t_0}^{t_0+\delta t_0} (f(x(t) + \delta x(t))\, dt + \int_{t_f}^{t_f+\delta t_f} (f(x(t) + \delta x(t))\, dt \\
&\quad + \int_{t_0}^{t_f} (f(x(t) + \delta x(t))\, dt - \int_{t_0}^{t_f} f(x(t))\, dt
\end{aligned}$$

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- $\int_{t_0}^{t_0+\delta t_0} f(x(t) + \delta x(t))\, dt \approx (f(x(t_0) + \delta x(t_0))\delta t_0 = -f(x(t_0))\, \delta t_0 + H.O.T$
- $\int_{t_f}^{t_f+\delta t_f} f(x(t) + \delta x(t))\, dt \approx (f(x(t_f) + \delta x(t_f))\delta t_f = f(x(t_f))\, \delta t_f + H.O.T$
- $\int_{t_0}^{t_f} f(x(t) + \delta x(t))\, dt - \int_{t_0}^{t_f} f(x(t))\, dt = \int_{t_0}^{t_f} \big(f(x(t) + \delta x(t)) - f(x(t)\big)\, dt$
  $= \int_{t_0}^{t_f} \big(f(x(t) + \frac{\partial f(x(t))}{\partial x(t)} \cdot \delta x + H.O.T - f(x(t)\big)dt \approx \int_{t_0}^{t_f} \frac{\partial f(x(t))}{\partial x(t)} \cdot \delta x\, dt$

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$$\boxed{\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} \frac{\partial f(x(t))}{\partial x(t)} \cdot \delta x dt + f(x(t_f))\delta t_f - f(x(t_0))\delta t_0}$$

The variation of a functional: example 2

How to compute variation of $J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt$ for fixed t_0 (assuming f has first and second continuous derivative)?

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} (g_x - \frac{d}{dt} g_{\dot{x}}) \cdot \delta x dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\delta x(t_f) + g(x(t_f), \dot{x}(t_f), t_f)\delta t_f$$

($g_x = \frac{\partial g}{\partial x}$, $g_{\dot{x}} = \frac{\partial g}{\partial \dot{x}}$)

See next page for the derivation

The variation of a functional: example 2 (cont'd)

$$\Delta J(x(t),\delta x(t)) = J(x(t)+\delta x(t)) - J(x(t))$$
$$= \int_{t_0}^{t_f+\delta t_f} (g(x(t)+\delta x(t), \dot{x}(t)+\delta\dot{x}(t), t)\,dt - \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt$$
$$= \int_{t_f}^{t_f+\delta t_f} (g(x(t)+\delta x(t), \dot{x}(t)+\delta\dot{x}(t), t)\,dt$$
$$+ \int_{t_0}^{t_f} (g(x(t)+\delta x(t), \dot{x}(t)+\delta\dot{x}(t), t)\,dt - \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt$$

$\star$ $\int_{t_f}^{t_f+\delta t_f} (g(x(t)+\delta x(t), \dot{x}(t)+\delta\dot{x}(t), t)\,dt \approx (g(x(t_f)+\delta x(t_f), \dot{x}(t_f)+\delta\dot{x}(t_f), t_f)\delta t_f =$
$g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f + \text{H.O.T}$

$\star$ $\int_{t_0}^{t_f} (g(x(t)+\delta x(t), \dot{x}(t)+\delta\dot{x}(t), t)\,dt - \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt$

$= \int_{t_0}^{t_f} \left(g_x(x(t),\dot{x}(t),t)\,\delta x(t) + g_{\dot{x}}(x(t),\dot{x}(t),t)\,\delta\dot{x}(t)\right) dt$

Let $u = g_{\dot{x}}$, **and** $d v = \delta\dot{x}\,\mathbf{d}t$ **to get:**

$\delta\dot{x} = \frac{d}{dt}\delta x$ ($\delta\dot{x}$ **and** δx **are not independent**)

Integration by parts: $\int_a^b u\,\mathbf{d}v = uv\Big|_a^b - \int_a^b v\,\mathbf{d}u.$

$= \int_{t_0}^{t_f} g_x(x(t),\dot{x}(t),t)\,\delta x(t)\,\mathbf{d}t -$
$\int_{t_0}^{t_f} \left(\frac{d}{dt} g_{\dot{x}}(x(t),\dot{x}(t),t)\,\delta x(t)\,dt + g_{\dot{x}}(x(t),\dot{x}(t),t)\Big|_{t=0}^{t_f}\right.$

$\delta J(x(t),\delta x(t)) = g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f + g_{\dot{x}}(x(t_f),\dot{x}(t_f),t_f)\,\delta x(t_f) - \overbrace{g_{\dot{x}}(x(t_0),\dot{x}(t_0),t_0)\,\delta x(t_0)}^{\delta x(t_0)=0,\textbf{fixed and given initial condition}} \to 0$ $+$

$\int_{t_0}^{t_f} \left(g_x(x(t),\dot{x}(t),t) - \frac{d}{dt} g_{\dot{x}}(x(t),\dot{x}(t),t)\right)\delta x(t)\,dt$

Extremal of a functional: fundamental theorem of the calculus of variation

Minimizer of a function $f(q)$ is $q^\star$ if

$$f(q^\star) \leqslant f(q)$$

for all admissible q in $\|q - q^\star\| \leqslant \epsilon$

Minimizer of a functional $J(x(t))$ is $x^\star(t)$ if

$$J(x^\star(t)) \leqslant J(x(t))$$

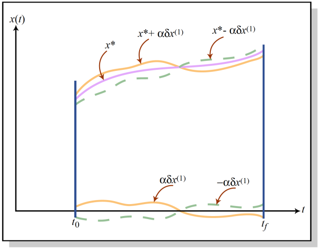
for all admissible $x(t)$ in $\|x(t) - x^\star(t)\| \leqslant \epsilon$.

Fundamental theorem of the calculus of variation

- Let x be a vector function of t in the class Ω, and $J(x)$ be a differential functional of x.
- Assume that all $x \in \Omega$ are not constrained by any boundaries. If $x^\star$ is an extremal function, the variation of J must vanish in $x^\star$

$$\delta J(x^\star, \delta x) = 0$$

for all admissible $x \in \Omega$.



Optimal control problems of interest

We are going to study

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t)dt), \quad \text{s.t.}$$
$$\dot{x}(t) = f(x(t),u(t),t),$$
$$x(t_0),\ t_0 \text{ is given},$$
$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$
$$x(t):\mathbb{R}\to\mathbb{R}^n,\quad u(t):\mathbb{R}\to\mathbb{R}^m,\quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n.$$

We will first focus on the special case below.[1]

$$x^\star(t)\Big|_{t\in[t_0,t_f]} = \text{argmin}\left(J(x(t)) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt\right) \text{ s.t.}$$
$$x(t_0) = x_0,$$
$$x(t_f) = x_f \quad (\text{various terminal conditions })$$

[1] "**Think of it as a case that we can find $u(t)$ in terms of $(x(t),\dot{x}(t))$ from $\dot{x}(t) = f(x(t),u(t),t)$. Then the optimal control problem above can be cast with $u(t)$ eliminated.**"

First order optimality conditions

$$x^\star(t)\Big|_{t\in[t_0,t_f]} = \text{argmin}\left(J(x(t)) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt\right) \text{ s.t.}$$
$$x(t_0) = x_0,$$
$$x(t_f) = x_f \quad (\text{various terminal conditions })$$

Variation

$$\delta J(x(t),\delta x(t)) = g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f + g_{\dot{x}}(x(t_f),\dot{x}(t_f),t_f)\,\delta x(t_f) + \int_{t_0}^{t_f}\left(g_x(x(t),\dot{x}(t),t) - \frac{d}{dt} g_{\dot{x}}(x(t),\dot{x}(t),t)\right)\delta x(t)\,dt$$

From this variation, for different terminal conditions, we are going to derive first order necessary conditions for optimality using the Fundamental Theorem of Calculus of Variation.

- Both t_f and $x(t_f)$ are specified and are given
  - In this case $\delta t_f = 0$ and $\delta x(t_f) = 0$
  - $\delta J(x(t),\delta x(t)) = \int_{t_0}^{t_f}\left(g_x(x(t),\dot{x}(t),t) - \frac{d}{dt} g_{\dot{x}}(x(t),\dot{x}(t),t)\right)\delta x(t)\,dt = 0 \Rightarrow$

the (first order) necessary condition for a maximum or minimum is called Euler Equation

$$g_x(x(t),\dot{x}(t),t) - \frac{d}{dt} g_{\dot{x}}(x(t),\dot{x}(t),t) = 0$$

In this case we solve the Euler Equation with the boundary conditions $x(0) = x_0$ and