Homework Assignment 3
Optimal Control- MAE 274
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Turn in your HW electronically to the respective folder in Canvas. You do not need to return your codes, only include the codes with your written HW.

Problem 1. Do problem 2.3.1 from Ref. [2]. This reference is available at UCI library as an e-book. Also you can obtain a copy from the link below
http://www.uta.edu/utari/acs/FL%20books/Lewis%20optimal%20control%203rd%20edition%202012.pdf
To discretize your system, use the method described in section 2.3 of Ref[2], see page 53.

Problem 2. Consider the following system

$$x(k+1) = \begin{pmatrix} 1.1 & 2 \\ 0 & 1.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k), \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) (pre-specified final state) Design a minimum energy controller to take this system from its given initial state to $x(N) = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$ (show your work) where N=10. Plot the system trajectories and the control history vs. timestep k.

(b) (Free final state) Consider the performance measure below

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k.$$

- Find an optimal controller which minimizes J for N=10 and N=100 for
$$Q = \begin{pmatrix} 1 & -5 \\ -5 & 25 \end{pmatrix}, \qquad R = 6, \qquad S_N = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$
Plot the state trajectory and control history of your system under optimal controllers you obtain.
- Compute the steady state ARE solution S_∞ *by* the two methods below:
  - Iterating the dynamic Riccati equation backward for a large k;
  - Using the Matlab command dare (to setup the inputs to this Matlab command to match the ARE that you have study the description of this command at http://www.mathworks.com/help/control/ref/dare.html

  Does steady state solution exist? Are your solutions the same? Compute the steady state feedback gain K_∞ . Is K_∞ an asymptotically stabilizing controller. Use Theorems 2.4.1 and 2.4.2 of Ref [2] to explain your observation.

- Choose your own S_N, Q, R, such that the steady state stabilizing controller exists (consult Theorem 2.4.2 of Ref[2]).
- Repeat the parts above for your choice of (S_N, Q, R). Continue the rest of the problem using your own (S_N, Q, R).
  - Compute the steady state feedback gain K_∞ .
  - Plot the state trajectories for both cases of N=10, N=100 horizon under optimal and steady state controller.
  - For the given initial conditions compute your cost under optimal and steady state controller for both cases of N=10, N=100.

- Compute the steady state solution of ARE equation, S_∞, (using any valid method you choose) for the same values of Q, R as your choice earlier but two different values of S_N. Are your solutions the same?

Problem 3.

a) Consider a LTI discrete-time system (A,C,B). Let (A,C) be observable. Show that $(A - BK, \begin{bmatrix} C \\ DK \end{bmatrix})$ is observable, where D is a square and invertible matrix with appropriate dimensions.

b) (do not turn in) Study the proof of Theorem 2.4.2 of Ref[2]. In particular think about how the observation that you make in part (a) helps to complete the proof of this theorem.

- If $X > 0$, then to obtain C that satisfies $X = C^T C$, you can use Matlab command chol, see the description at http://www.mathworks.com/help/matlab/ref/chol.html

- If $X \geq 0$, then to obtain C that satisfies $X = C^T C$, you can use the following Matlab commands

```
r = rank(X);
[U S V]=svd(X);
C=sqrt(S(1:r,1:r))*U(:,1:r)'
```

  This method can also be used for $X > 0$.

- In your Matlab codes, after computing S_k at each backward iteration of the Riccati equation, use $S_k = (S_k + S_k^T)/2$ to make sure that your S_k stays symmetric. The symmetry can be lost due to accumulated numerical errors.

- A linear discrete-time system described by the state equation $x(k+1) = A_{cl}x(k)$ is asymptotically stable if and only if all eigenvalues have magnitude smaller than one, i.e, its eigenvalues are inside the unit ball centered in the origin of the complex plane.

- Recall that the Lyapunov stability theorem for discrete time system $x(k+1) = A_{cl}x(k)$ states that
  - $x(k+1) = A_{cl}x(k)$ is asymptotically stable if and only if there exists a matrix $P > 0$ that satisfies $P = A_{cl}^T P A_{cl} + Q$ for any real $Q > 0$.
  - Let (A_{cl}, C) be observable. Then, $x(k+1) = A_{cl}x(k)$ is asymptotically stable if and only if there exists a matrix $P > 0$ that satisfies $P = A_{cl}^T P A_{cl} + C^T C$.