MAE270A: Concepts of Observability/Detectability for LTI Systems
Observer design for LTI systems
Observer-based state feedback design for LTI systems

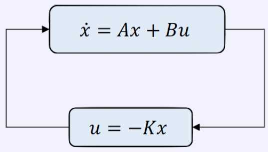
Instructor: Solmaz Kia (solmaz@uci.edu), University of California Irvine

Problem setting.

Consider the linear time invariant system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p,\ y \in \mathbb{R}^q.$$

Previously, we saw that when (A, B) is stabilizable, the full-state feedback can be used to stabilize the system.

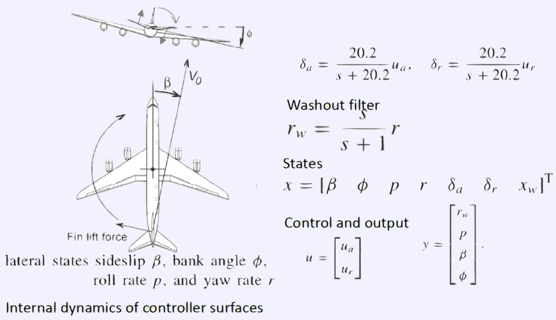


To stabilize the system using full-state feedback, we must be able to access all of the system's states. Unfortunately, this is not always possible.

Why can't we access all states.

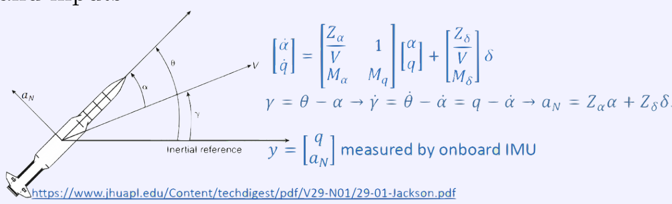
There are various reasons that we may not have access to all states:

- Cost: we access the system states through sensors. Sensors can be expensive, and cost may limit what states we can measure
- You may not have a sensor to measure every state



In this example, there is no sensor to measure x_w. Also, the sensor that we have measures r_w rather than r.

- Sensors you have measure combination of the states and inputs



States and control inputs are, respectively, $x = \begin{bmatrix} \alpha \\ q \end{bmatrix}$ and $u = \delta$. Thus, $C = \begin{bmatrix} 0 & 1 \\ Z_\alpha & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 \\ Z_\delta \end{bmatrix}$. The acceleration measured by the IMU is a linear combination of a state and a control input.

Definition: Observability

Observability refers to determining $x(0)$ from the future inputs and outputs $u(t)$ and $y(t)$, $t \in [0, T]$ for any finite $T \in \mathbb{R}_{>0}$.

Question of interest in Observability.

Can we reconstruct $x(0)$ by knowing $y(t)$ and $u(t)$ over some finite time interval $[0, T]$? We can use the initial condition to calculate the entire state $x(t)$ by solving the differential equation $\dot{x} = Ax + Bu$, and then use it in our state feedback to control the system.

Observability Gramian.

Some analysis:

$$y(t) = C\mathrm{e}^{A\,t}x(0) + C\int_0^t \mathrm{e}^{A\,(t-\tau)}Bu(\tau)\mathrm{d}\tau.$$

We assume that we have access to $y(t)$ and $u(t)$:

$$\underbrace{y(t) - C\int_0^t \mathrm{e}^{A\,(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{\bar{y}(t):\ \text{known}} = C\,\mathrm{e}^{A\,t}\underbrace{x(0)}_{\text{unknown}}.$$

Let's multiply both sides of the equality above $\mathrm{e}^{A^\top t}C^\top$:

$$\underbrace{\mathrm{e}^{A^\top t}C^\top\,\bar{y}(t)}_{\mathbb{R}^n} = \underbrace{\mathrm{e}^{A^\top t}C^\top\,C\mathrm{e}^{At}}_{\mathbb{R}^{n\times n}}\underbrace{x(0)}_{\mathbb{R}^n}.$$

Next, integrate both size over the finite time $[0, t]$:

$$\underbrace{\int_0^t \mathrm{e}^{A^\top\tau}C^\top\,\bar{y}(\tau)\mathrm{d}\tau}_{\text{known}} = W_O(t)x(0)$$

where

$$\text{Observability Gramian}: \ W_O(t) = \int_0^t \mathrm{e}^{A^\top\tau}C^\top\,C\mathrm{e}^{A\tau}\mathrm{d}\tau. \quad (1)$$

- Condition for observable system: unique $x(0)$ can be obtained

$$\mathrm{Rank}(W_O(t)) = n.$$

Unobservable system: a unique $x(0)$ cannot be obtained

$$\mathrm{Rank}(W_O(t)) < n.$$

Unobservable subspace.

Unobservable subspace of system (A, C) is $\mathrm{Ker}(W_O(t))$:

$$\text{If } x(0) \in \mathrm{Ker}(W_O(t)), \text{ then } W_O(t)x(0) = 0.$$

Controllability/Observability duality.

Recall the controllability Gramian for system (A, B):

$$W_c(t) = \int_0^t \mathrm{e}^{A\tau} B B^\top \mathrm{e}^{A^\top \tau} \mathrm{d}\tau.$$

Consider a fictitious system $(A^\top, C^\top)$ and form the controllability Gramian for this system

$$W_c(t) = \int_0^t \mathrm{e}^{A^\top \tau} C^\top C \mathrm{e}^{A\tau} \mathrm{d}\tau. \tag{2}$$

Comparing the observability Gramian (1) with the controllability Gramian (2), note that the observability Gramian for (A, C) is identical to the controllability Gramian for $(A^\top, C^\top)$.

Duality: (A, C) is observable if and only if $(A^\top, C^\top)$ is controllable.

Duality allows to recycle the conditions derived controllability and state feedback design for observability and observer design. For example, $(A^\top, C^\top)$ is controllable if and only if

$$\text{Rank} \begin{bmatrix} C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1} C^\top \end{bmatrix} = n.$$

This condition is equivalent to Rank$\mathcal{O} = n$ where

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3}$$

By virtue of duality condition, then (A, C) is observable if and only if Rank $\mathcal{O} = n$. As such matrix $\mathcal{O}$ is called observability matrix.

Conditions for Observability

The n-dimentional pair (A, C) is equivalent to either of the conditions below

- The $\mathbb{R}^{n\times n} \ni W_O(t) = \int_0^t \mathrm{e}^{A^\top \tau} C^\top C \mathrm{e}^{A\tau} \mathrm{d}\tau$ is nonsingular for all $t \in \mathbb{R}_{>0}$.
- Rank $\mathcal{O} = n$, where $\mathcal{O}$ is the observability matrix given in (3).
- Rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$, for any complex number λ.
- Rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$, for every eigenvalue λ of A.
- If in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$A^\top W_O + W_O A = -C^\top C$$

is positive definite. The solution is called the observability Gramian and can be expressed as

$$W_O = \int_0^\infty \mathrm{e}^{A^\top \tau} C^\top C \mathrm{e}^{A\tau} \mathrm{d}\tau.$$

Unobservable LTI system: Observable decomposition.

Suppose Rank$\mathcal{O} = m \leq n$. This means that (A, C) is not observable. There exists invertible T s.t. $\bar{x} = T^{-1}x$ transforms state equations to $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\,u$ and $y = \bar{C}\bar{x} + D\,u$, where

$$\bar{A} = \underbrace{\begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}}_{T^{-1}AT}, \quad \bar{B} = \underbrace{\begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}}_{T^{-1}B}, \quad \bar{C} = \underbrace{\begin{bmatrix} C_o & 0_{q\times(n-m)} \end{bmatrix}}_{CT}, \qquad (4)$$

where $A_o \in \mathbb{R}^{m\times m}$, $B_o \in \mathbb{R}^{m\times p}$ and $C_o \in \mathbb{R}^{q\times m}$.

The similarity transformation matrix T is

$$T = \Big[\ \underbrace{t_1 \ \ t_2 \ \cdots \ t_m}_{} \quad | \quad \underbrace{t_{m+1} \ \cdots \ t_n}_{}\ \Big].$$

- m vectors whatever way that makes all columns of T linearly independent
- $n - m$ linearly independent vectors that spans the nullspace of $\mathcal{O}$

- (A_o, C_o) is observable.
- The transfer function of the system (A, B, C, D) is $G(s) = C(sI - A)^{-1}B + D$. For an unobservable system

$$G(s) = C_o(sI - A_o)^{-1}B_o + D.$$

Detectablility.

Recall the observable decomposition (4), where $\bar{x} = (\bar{x}_o, \bar{x}_{\bar{o}})$ with $\bar{x}_o \in \mathbb{R}^{m\times m}$ and $\bar{x}_{\bar{o}} \in \mathbb{R}^{(n-m)\times(n-m)}$. The observable decomposition reads as

$$\begin{cases} \dot{\bar{x}}_o = A_{\bar{o}}\bar{x}_o, \\ \dot{\bar{x}}_{\bar{o}} = A_{\bar{o}}\bar{x}_{\bar{o}} + A_{21}\bar{x}_o + B_{\bar{o}}\,u. \end{cases}$$

- Because (A_o, C_o) is observable, we can construct $\bar{x}_o(0)$ and consequently $\bar{x}_o(t)$ in finite time from knowing $y(t)$ and $u(t)$.

$$\bar{x}_{\bar{o}}(t) = \mathrm{e}^{A_{\bar{o}}\,t}\underbrace{\bar{x}_{\bar{o}}(0)}_{\text{unknown}} + \underbrace{\int_0^t \mathrm{e}^{A_{\bar{o}}\,(t-\tau)}(A_{21}\bar{x}_o(\tau) + B_{\bar{o}}\,u(\tau))\mathrm{d}\tau}_{\text{known}}.$$

When $A_{\bar{o}}$ is Hurwitz, we have $\mathrm{e}^{\bar{A}_{\bar{o}}\,t} \to 0$ as $t \to \infty$. This means

- $\bar{x}_{\bar{o}}$ can be guessed to an error that converges to zero exponentially fast.

Therefore we can have $\bar{x}$ and x detectable as $t \to \infty$. On the other hand,

Unobservable eigenvalues.

Suppose (A, C) is unobservable. The 'unobservable eigenvalues' of A are

- the eigenvalue that fail the PBH test.
- eigenvalues of $A_{\bar{o}}$.

Detectability condition

The pair (A, C) is detectable if it is observable. If (A, C) is unobservanle, matrix $A_{\bar{o}}$ is Hurwitz or alternatively, all 'unobservable eigenvalues' of A have negative real parts.

State observer design.

We start with an observation. We don't know $x(0)$, but we know the system dynamics, so let's see if we create a virtual system

$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + Bu \\ y = C\,\hat{x} + Du \end{cases}$$

and initialize with $\hat{x}(0)$, which is not the same as $x(0)$, what happens. Let $e(t) = \hat{x}(t) - x(t)$. This error state evolve according to

$$\dot{e} = A\hat{x} + Bu - (A\hat{x} + Bu) \rightarrow \dot{e} = A\,e, \quad e(0) = \hat{x}(0) - x(0).$$

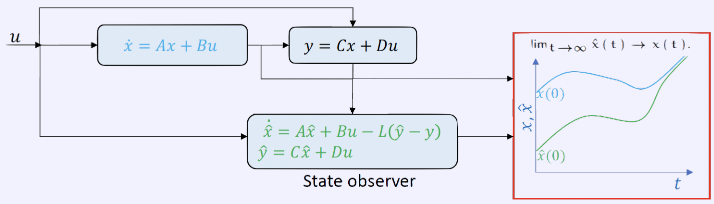
- If A is Hurwitz $e \to 0$ as $t \to \infty$. This means that as $t \to \infty$ (for some $t > T$) we have $\hat{x}(t) \to x(t)$ exponentially.
- Otherwise, $e(0) \neq 0$ will not result to $e(t) \neq 0$ after some $t > T$.

Research question.

- How to drive $e \to 0$ as $t \to \infty$ if A is not Hurwitz.
- Even if A is Hurwitz, the rate of convergence of $e \to 0$ may be very slow. How to speed up the convergence?

Solution.

Use $\hat{y} - y$ as a feedback to guide e to zero and manage the speed of convergence.



$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u - L\,(\hat{y} - y) \\ y = C\,\hat{x} + Du \end{cases}, \tag{5}$$

where L is <u>output injection</u> matrix.

The error dynamics using (5) is

$$\dot{e} = (A - CL)\,e$$

- Choose L such that $A - CL$ is Hurwitz
- Choose L to place the eigenvalues of $A - CL$ in places that will result in desired exponential convergence rate for $e \to 0$.

Research question.

- When does a stabilizing L, i.e., L that makes $A - CL$ Hurwitz exists.
- If yes, how to design L to place eigenvalues of $A - CL$ at desired location $\{\nu_1, \cdots, \nu_n\}$.

Take away for state observer design

- If (A, C) is observable, there exists an $L \in \mathbb{R}^{n\times q}$ that can place the eigenvalue of $A - LC$ at any desired (symmetric complex pair) locations $\{\nu_1, \cdots, \nu_n\}$.
- For a detectable (A, C) let $\{\lambda_1, \cdots, \lambda_m\}$ be the observable eigenvalues of A and $\{\lambda_{m+1}, \cdots, \lambda_n\}$ be the observable eigenvalues of A. If (A, C) is detectable, then there exists an $L \in \mathbb{R}^{n\times q}$ that can place the eigenvalues of $A - LC$ at $\{\nu_1, \cdots, \nu_m, \lambda_{m+1}, \cdots, \lambda_n\}$, where $\{\nu_1, \cdots, \nu_m\}$ are the desired (symmetric complex pair) eigenvalue locations for moving the observable eigenvalue $\{\lambda_1, \cdots, \lambda_m\}$.

Observation.

$$\text{eig}(A - LC) = \text{eig}(A^\top - C^\top L^\top).$$

Thus, if you design $L^\top$ to place eigenvalues of $(A^\top - C^\top L^\top)$ at desired locations, it is equivalent to design L to place eigenvalues of $(A - LC)$ at those desired locations.

State observer design.

- Duality says that if (A, C) is observable, then $(A^\top, C^\top)$ is controllable.
- Recall the stabilizing state feedback control gain design that made $A - BK$. The same methods can be used to design $L^\top$ to make $(A^\top - C^\top L^\top)$ Hurwitz or use the pole-placement methods to design $L^\top$ to place the eigenvalues of $(A^\top - C^\top L^\top)$.

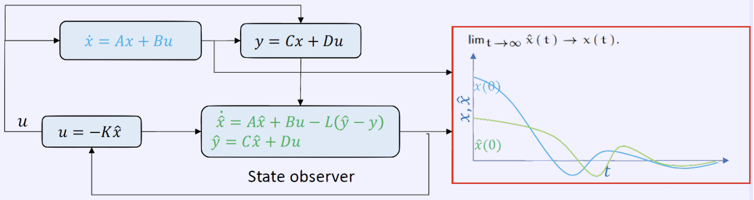
- Duality says that if (A, C) is detectable, then $(A^\top, C^\top)$ is stabilizable.
- Recall the stabilizing state feedback control gain design that made $A - BK$ when (A, B) is stabilizable. The same methods can be used to design $L^\top$ to make $(A^\top - C^\top L^\top)$ Hurwitz or use the pole-placement methods to design $L^\top$ to place the eigenvalues of $(A^\top - C^\top L^\top)$ at $\{\nu_1, \cdots, \nu_m, \lambda_{m+1}, \cdots, \lambda_n\}$.

Research question.

- When we do not have access to all the states x, can we use the output y to stabilize the system?

Stabilization using output feedback.

We want to stabilize the system using asymptotically constructed states, $\hat{x}$. That is we want to use $u = -K\hat{x}$ in the architecture below:



$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases} \qquad \begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u \\ y = C\,\hat{x} + D\,u \end{cases} \qquad u = -K\,\hat{x}. \tag{6}$$

How to design K.

To study whether the system (6) with output feedback $u = -K\,\hat{x}$ is exponentially stable, form the close-loop system matrix with states x and $e = \hat{x} - x$:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - B\,K & -B\,K \\ 0 & A - L\,C \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ e \end{bmatrix}. \tag{7}$$

Note that since $\bar{A}$ in block triangular matrix, we have

$$\text{eig}(\bar{A}) = \text{eig}(A - B\,K) \cup \text{eig}(A - L\,C). \tag{8}$$

- For $\bar{A}$ to be Hurwitz, we need to design K and L such that all the eigenvalues of $\bar{A}$ have negative real parts.
- The separation on display in (8) presents a systematic design for making $\bar{A}$ Hurwitz:
  - Design K for stabilizing $A - B\,K$ (using methods for full state feedback design)
  - Design L for stabilizing $A - L\,C$ (using the methods for state observer design)

Last observation.

Note that

$$\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}}_{\bar{T}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

Because $\bar{T}$ is invertible, $\bar{T}$ is a similarity transformation matrix. Therefore, the closed-loop system with states (x, e) is algebraically equivalent to the closed-loop system with states $(x, \hat{x})$. If the pair (L, K) stabilizes the closed-loop system with states (x, e), it will stabilize $(x, \hat{x})$.

References.

- Joao P. Hespanha. Linear Systems Theory, 2009
- C. T. Chen. Linear System Theory and Design, 4th edition.
- Franklin and Powell. Feedback Control