Linear Systems I
Lecture 6

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Summary of previous lecture and today's otutline

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

$\phi(t,\tau) = e^{A(t-\tau)}$

Theorem (Variation of constants): The unique solution to [illegible] equation above is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t),$$

where $\phi(t, t_0)$ is the state transition matrix (as defined before).

$$y(t) = \underbrace{C(t)\phi(t, t_0)x_0}_{\text{homogeneous response}} + \underbrace{\int_{t_0}^{t} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\text{forced response}}.$$

How to compute e^{At}:

- The ith column of e^{At} is the solution of $\dot{x} = Ax,\ x(0) = e_i$
- $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$

Def (Algebraically equivalent) Two continuous-time LTI systems

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t),\\ y(t) = Cx(t) + Du(t),\end{cases} \quad \text{and} \quad \begin{cases}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t),\\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t),\end{cases}$$

are called algebraically equivalent if and only if there exists a nonsingular T s. t. ($\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$). The corresponding map $\bar{x} = Tx$ is called a similarity transformation or an equivalence transformation.

A and $\bar{A}$ have same eigenvalues.

Lecture 6 covers

- Review of eigenvalues and eigenvectors of a matrix
- Jordan form of a matrix
- Use of Jordan/diagonalized form to compute e^{At}

Review of eigenvalues and eigenvectors of a matrix

Consider a matrix $A \in \mathbb{R}^{n\times n}$,

$$Ap = \lambda p,$$

- $\lambda \in \mathbb{C}$ is eigenvalue iff we have $p \in \mathbb{C}^{n\times 1}$, $p \neq 0_{n\times 1}$

- Compute λ: $\Delta(A) = \det(\lambda I - A) = 0$; has n roots $\Rightarrow$ n eigenvalues

- Computing eigenvectors: $q \neq 0$ such that $(\lambda I - A)p = 0$, i.e., q is in the nullspace of $(\lambda I - A)$,
- Some of the properties of the eigenvectors
  - When all the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a $n \times n$ matrix A are distinct (multiplicity of all eigenvalues is 1), the nullity of $(\lambda_i I - A)$ is equal to 1. Moreover, the corresponding eigenvector set $\{p_1, \cdots, p_n\}$ is linearly independent.
  - When $\bar{\lambda}$ is an eigenvalue of A with multiplicity of $m \in [2, n]$, then we have $1 \leqslant \text{nullity}(\bar{\lambda} I - A) \leqslant m$.

Diagonalizable matrix

If A has only simple eigenvalues, it always has a diagonal form representation, i.e., there exists Q such that

$$\mathrm{J} = Q\,A\,Q^{-1}$$

$Q = P^{-1}$

$$A\,p_i = \lambda_i\,p_i, i \in \{1,\cdots,n\} \rightarrow A\underbrace{[p_1 \quad \cdots \quad p_n]}_{P} = [p_1 \quad \cdots \quad p_n]\underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}}_{J}$$

$$A = P\,\mathrm{J}\,P^{-1}$$

When all the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a $n \times n$ matrix A are distinct (multiplicity of all eigenvalues is 1), the nullity of $(\lambda_i I - A)$ is equal to 1. Moreover, the corresponding eigenvector set $\{p_1, \cdots, p_n\}$ is linearly independent.

An eigenvalue with multiplicity of 2 or higher is called a **repeated eigenvalue**.

In contrast, an eigenvalue with multiplicity of 1 is called a **simple eigenvalue**.

If A has a repeated eigenvalues, then it may not have a diagonal form representation. However, it has a block-diagonal and triangular form representation.

Diagonalizable matrix

If A has only simple eigenvalues, it always has a diagonal form representation, i.e., there exists Q such that

$$J = QAQ^{-1}$$

$$Q = P^{-1}$$

$$Aq_i = \lambda_i q_i, i \in \{1, \cdots, n\} \to A[p_1 \quad \cdots \quad p_n] = [p_1 \quad \cdots \quad p_n]\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

$$A = P\, J\, P^{-1}$$

When all the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a $n \times n$ matrix A are distinct (multiplicity of all eigenvalues is 1), the nullity of $(\lambda_i I - A)$ is equal to 1. Moreover, the corresponding eigenvector set $\{p_1, \cdots, p_n\}$ is linearly independent.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(A) = \det(\lambda I - A) = 0$$

$$\Delta(A) = \lambda(\lambda + 1)(\lambda - 2) = 0$$

— — — — — — — — — — — —

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 0: \ (A - 0I)p_2 = 0,$

$\lambda_3 = 2: \ (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$P = \left[\begin{array}{c|c|c} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$$A = P \underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

— — — — — — — — — — — —

λ's are district

● the matrix is diagonalizable

Jordan normal form

Theorem(Jordan normal form): For every matrix $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular change of basis $\mathbf{Q} \in \mathbb{C}^{n\times n}$ that transforms A into

$$J = QAQ^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_l \end{bmatrix} = \text{Diag}(J_1, J_2, J_3, \cdots, J_l),$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

- **For every eigenvalue λ_i of A, there is at least one Jordan block associated with**
- **The number of Jordan block associated with each λ_i of A is equal to the nullity of $(A - \lambda_i I)$.**
- **If λ_j is an eigenvalue with multiplicity of $m_j = 1$, the Jordan block associated with it is $J_j = \lambda_j$**

Jordan normal form

Theorem(Jordan normal form): For every matrix $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular change of basis $\mathbf{Q} \in \mathbb{C}^{n\times n}$ that transforms A into

$$J = QAQ^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_l \end{bmatrix} = \text{Diag}(J_1, J_2, J_3, \cdots, J_l),$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

Attention: There can be several Jordan blocks for the same eigenvalue, but in that case there must be more than one independent eigenvector for that eigenvalue.

- λ_i is an eigenvalue of A
- l, number of Jordan blocks: total number of linearly independent eigenvectors of A
- J is unique up to a reordering of the Jordan blocks
- J is called Jordan normal form of A

Diagonalizable matrix

- An eigenvalue with multiplicity of 2 or higher is called a repeated eigenvalue.
- In contrast, an eigenvalue with multiplicity of 1 is called a simple eigenvalue.
- If A has only simple eigenvalues, it always has a diagonal form representation.
- If A has a repeated eigenvalues, then it may not have a diagonal form representation. However, it has a block-diagonal and triangular form representation.

Def. (Semisimple) A matrix is called semi-simple or diagonalizable if its Jordan normal form is diagonal.

— — — — — — — — — — — — — — —

Theorem Fo an $n \times n$ matrix A, the following statements are equivalent:

- A is semi-simple.
- A has n linearly independent eigenvectors.
- For any λ_i of A with multiplicity of m_i, we have nullity$(\lambda_i I - A) = m_i$.

— — — — — — — — — — — — — — —

One of the methods to determining the Jordan normal form

1. Compute eigenvalues of A
2. List all possible Jordan normal forms that are compatible with the eigenvalues of A:
   - eigenvalues with multiplicity equal to 1 must always correspond to 1×1 Jordan blocks
   - eigenvalues with multiplicity equal to 2 can correspond to one 2×2 block or two 1×1 blocks
   - eigenvalues with multiplicity equal to 3 can correspond to one 3×3 block , one 2×2 and two 1×1 blocks, or three 1×1 blocks, etc.
3. For each candidate Jordan normal form, check wether there exists a nonsingular matrix Q for which $\mathrm{J} = \mathrm{Q}A\mathrm{Q}^{-1}$. To find out wether this is so, you may solve the (equivalent, but simpler) linear equation

$$\mathrm{JQ} = \mathrm{Q}A$$

for the unknown matrix Q and check wether it has a nonsingular solutions.

Jordan normal form

A a 5×5 matrix with a simple eigenvalue λ_1, and λ_2 with multiplicity of $m = 4$

$$\exists \text{ invertible } Q: \qquad J = Q^{-1}AQ$$

$$J = \left[\begin{array}{c|c|c|c|c} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 4$

$$J = \left[\begin{array}{c|cc|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 2$

$$J = \left[\begin{array}{c|cccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 1$

$$J = \left[\begin{array}{c|c|c|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 3$

$$J = \left[\begin{array}{c|c|ccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 2$

Diagonal Jordan form: example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(A) = \det(\lambda I - A) = 0$$
$$\Delta(A) = \lambda(\lambda+1)(\lambda-2) = 0$$

– – – – – – – – – – – – – – –

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 0: \ (A - 0I)p_2 = 0,$

$\lambda_3 = 2: \ (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = P \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{J \text{ with } Q = P^{-1}} P^{-1}$$

– – – – – – – – – – – – – – –

λ's are district

- the matrix is diagonalizable
- the Jordan form is a diagonal matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(A) = \det(\lambda I - A) = 0$$
$$\Delta(A) = (\lambda+1)(\lambda-2)^2 = 0$$

– – – – – – – – – – – – – – –

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$$\begin{cases} \lambda_2 = 2, \text{ with } m_2 = 2, \\ \text{note that nullity}(A - 2I) = 2, \text{ therefore} \\ \text{two linearly independent eigenvectors exist for } \lambda_2: \\ (A-2I)p_2 = 0, \quad (A-2I)p_3 = 0, \end{cases}$$

linearly independent $\{p_1, p_2, p_3\}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A = P \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{J \text{ with } Q = P^{-1}} P^{-1}$$

– – – – – – – – – – – – – – –

Recall that

- The number of Jordan block associated with each λ_i of A is equal to the nullity of $(A - \lambda_i I)$.

if for every λ_i with multiplicity $m_i \geqslant 1$, we have nullity$(A - \lambda I) = m_i$

- the matrix is diagonalizable
- the Jordan form is a diagonal matrix

$$A = \begin{bmatrix} 3 & 1.5 & -2 \\ 0 & 2 & 0 \\ 1 & 1.25 & 0 \end{bmatrix}$$

$$\Delta(A) = \det(\lambda I - A) = 0$$
$$\Delta(A) = (s-2)^2(s-1) = 0$$

$$\lambda_1 = 1: \quad (A - I)p_1 = 0.$$

$$\begin{cases} \lambda_2 = 2, \text{with multiplicity } m_2 = 2, \\ \text{nullity of } (A - 2I) = 1, therefore, only\ one \\ linearly\ independent\ eigenvector\ exists\ for \\ \lambda_2 = 2, \qquad (A - 2I)p_2 = 0 \end{cases}$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

To find Q that satisfies $J = QAQ^{-1}$, we solve

$$JQ = QA,$$

$$\begin{bmatrix} q_{11} & q_{12} & q_{31} \\ 2q_{21} + q_{31} & 2q_{22} + q_{32} & 2q_{32} + q_{33} \\ 2q_{31} & 2q_{32} & 2q_{33} \end{bmatrix}$$
$$= \begin{bmatrix} 3q_{11} + q_{13} & 1.5q_{11} + 2q_{12} + 1.25q_{13} & -2q_{11} \\ 3q_{21} + q_{23} & 1.5q_{21} + 2q_{22} + 1.25q_{23} & -2q_{21} \\ 3q_{31} + q_{33} & 1.5q_{31} + 2q_{32} + 1.25q_{33} & -2q_{31} \end{bmatrix}$$

which gives (solution is not unique)

$$Q = \begin{bmatrix} -04082 & -0.4082 & 0.8165 \\ 0.6727 & -1.0328 & -0.6727 \\ 0 & 0.1682 & 0 \end{bmatrix}$$

Matrix exponential of two algebraically equivalent matrix

- Let T be nonsingular
- Let $A = T\bar{A}T^{-1}$,

$$e^{At} = Te^{\bar{A}t}T^{-1}$$

Proof

$$A^k = \underbrace{AAA\cdots A}_{k\ \text{times}} = \underbrace{(T\bar{A}T^{-1})(T\bar{A}T^{-1})\cdots(T\bar{A}T^{-1})}_{k\ \text{times}} = T\bar{A}^kT^{-1}$$

$$e^{At} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = \sum_{k=0}^{\infty}\frac{t^k}{k!}T\bar{A}^kT^{-1} = T(\sum_{k=0}^{\infty}\frac{t^k}{k!}\bar{A}^k)T^{-1} = Te^{\bar{A}t}T^{-1}$$

How to compute e^{At} using the Jordan normal form of A

$$J = QAQ^{-1} \quad \Longleftrightarrow \quad A = Q^{-1}JQ,$$

$$A^k = \underbrace{AAA\cdots A}_{k \text{ times}} = \underbrace{Q^{-1}JQQ^{-1}JQ\cdots Q^{-1}JQ}_{k \text{ times}} = Q^{-1}J^kQ$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = Q^{-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}J^kQ =$$

$$Q^{-1}\begin{bmatrix} \sum_{k=0}^{\infty}\frac{t^k}{k!}J_1^k & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}J_2^k & 0 & \cdots & 0 \\ 0 & 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}J_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{k=0}^{\infty}\frac{t^k}{k!}J_l^k \end{bmatrix}Q$$

$$= Q^{-1}\begin{bmatrix} e^{J_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{J_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{J_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{J_l t} \end{bmatrix}Q$$

How to compute e^{At} using the Jordan normal form of A

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

Claim: $$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

How can we verify the claim made above?

How to compute e^{At} using the Jordan normal form of A

Verification: we show that e^{J_i} is the transition matrix of J_i ($e^{J_i} = \phi(t,0)$) by showing that it satisfies $\begin{cases} \frac{d}{dt}\phi(t,0) = J_i\phi(t,0) \\ \phi(0,0) = I. \text{ That is} \end{cases}$

- $e^{J_i 0} = I$ (this is trivially satisfied)
- $\frac{d}{dt}e^{J_i t} = J_i e^{J_i t}$

$$\frac{d}{dt}e^{\lambda_i t} = \frac{d}{dt}e^{J_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \lambda_i e^{\lambda_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} +$$

$$e^{\lambda_i t}\begin{bmatrix} 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ 0 & 0 & 0 & \cdots & \frac{t^{n_i-4}}{(n_i-4)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda_i e^{J_i t} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} e^{J_i t} = J_i e^{\lambda_i t}.$$

How to compute e^{At} using the Jordan normal form of A: examples

$$J = \left[\begin{array}{c|c|ccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] \Rightarrow e^{Jt} = \left[\begin{array}{c|c|ccc} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ \hline 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ \hline 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

$$J = \left[\begin{array}{c|cccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] \Rightarrow e^{Jt} = \left[\begin{array}{c|cccc} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ \hline 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} & \frac{t^3}{6}e^{\lambda_2 t} \\ 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

How to compute e^{At} using the Jordan normal form of A: examples

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(A) = \lambda(\lambda+1)(\lambda-2) = 0$$

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 0: \ (A - 0I)p_2 = 0,$

$\lambda_3 = 2: \ (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$A\underbrace{[\ p_1 \mid p_2 \mid p_3\ ]}_{P} = [\ p_1 \mid p_2 \mid p_3\ ]\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \left[\begin{array}{c|c|c} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right], \ P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A = P\underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

$$e^{At} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}}_{P}\left[\begin{array}{c|c|c} e^{-t} & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{2t} \end{array}\right]\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}}_{P^{-1}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{e^{2t}}{6} - \frac{2e^{-t}}{3} + \frac{1}{2} & \frac{2e^{-t}}{3} + \frac{e^{2t}}{3} & \frac{2e^{2t}}{3} - \frac{2e^{-t}}{3} \\ \frac{e^{-t}}{3} + \frac{e^{2t}}{6} - \frac{1}{2} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{-t}}{3} + \frac{2e^{2t}}{3} \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(A) = (\lambda+1)(\lambda-2)^2 = 0$$

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 2$, **with** $m_2 = 2$,
note that nullity$(A - 2I) = 2$, **therefore**
two linearly independent eigenvectors exist for λ_2 :
$(A - 2I)p_2 = 0, \quad (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$A\underbrace{[\ p_1 \mid p_2 \mid p_3\ ]}_{P} = [\ p_1 \mid p_2 \mid p_3\ ]\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = P\underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

$$e^{At} = \underbrace{\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{P}\left[\begin{array}{c|c|c} e^{-t} & 0 & 0 \\ \hline 0 & e^{2t} & 0 \\ \hline 0 & 0 & e^{2t} \end{array}\right]\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ e^{2t} - e^{-t} & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

References

[1] Joao P. Hespanha, ``Linear systems theory”, Princeton University Press (Chapter 7)