Linear Systems I
Lecture 5

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Summary of previous lecture and today's outline

We want to study the properties of solutions to SS LTV systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

$A(t) : [0, \infty) \to \mathbb{R}^{n\times n}$, $B(t) : [0, \infty) \to \mathbb{R}^{n\times p}$, $C(t) : [0, \infty) \to \mathbb{R}^{q\times n}$, $D(t) : [0, \infty) \to \mathbb{R}^{q\times p}$.

P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geqslant t_0.$$

P.2 For evert fixed t_0, the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geqslant t_0,$$

where e_i is the i^{th} column of identity matrix I_n, or equivalently a column vector of all zero entries except for the i^{th} which is equal to 1.

P3. For every t, s, τ we have

$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau).$$

This property is called the semigroup property.

P4. For evert t, τ, $\phi(t, t_0)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

Theorem (Variation of constants): The unique solution to LTV SS equation above is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t),$$

where $\phi(t, t_0)$ is the state transition matrix (as defined before).

$$y(t) = \underbrace{C(t)\phi(t, t_0)x_0}_{\text{homogeneous response}} + \underbrace{\int_{t_0}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\text{forced response}}.$$

Lecture 5 covers

- Solution of LTI systems
  - Properties of matrix exponential
  - Cayley-Hamilton Theorem
  - Methods to compute e^{At}

Solution of an LTI system

We want to study the properties of solutions to SS LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times p}$. Start by study of

homogeneous linear system: $\dot{x} = Ax(t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \; t \geqslant t_0. \quad (1)$

Theorem (Peano-Baker Series). The unique solution to (1) is given by

$$x(t) = \phi(t, t_0)x_0, \qquad (2)$$

$$\phi(t,t_0) = I + \int_{t_0}^{t} A d\tau_1 + \int_{t_0}^{t} A \int_{t_0}^{\tau_1} A d\tau_2 d\tau_1 + \int_{t_0}^{t} A \int_{t_0}^{\tau_1} A \int_{t_0}^{\tau_2} A d\tau_3 d\tau_2 d\tau_1 + \cdots$$

$$= I + A\int_{t_0}^{t} d\tau_1 + A^2 \int_{t_0}^{t}\int_{t_0}^{\tau_1} d\tau_2 d\tau_1 + A^3 \int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2} d\tau_3 d\tau_2 d\tau_1 + \cdots$$

$$= I + \frac{(t-t_0)}{1}A + \frac{(t-t_0)^2}{2!}A^2 + \frac{(t-t_0)^3}{3!}A^3 + \cdots + \frac{(t-t_0)^k}{k!}A^k + \cdots = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}A^k$$

$$\phi(t,t_0) = e^{A(t-t_0)}$$
$$e^{A(t-t_0)} = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k$$

Properties of the transition matrix of and LTI system

- P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geqslant t_0.$$

  - LTI: The function e^{At} is the unique solution of

$$\frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A0} = I, \quad t \geqslant 0.$$

- P.2 For evert fixed t_0, the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geqslant t_0.$$

  - LTI: The i^{th} column of e^{At} is the unique solution to

$$\dot{x} = Ax(t), \quad x(0) = e_i, \quad t \geqslant 0.$$

- P3. For every t, s, τ we have (semigroup property)

$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau).$$

  - LTI: For every $t, \tau \in \mathbb{R}$

$$e^{A\tau}e^{At} = e^{At}e^{A\tau} = e^{A(t+\tau)}. \quad \text{But in general} \quad e^{At}e^{Bt} \neq e^{(A+B)t}.$$

- P4. For evert t, τ, $\phi(t, \tau)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

  - LTI: For every $t \in \mathbb{R}$, the function e^{At} is nonsingular and

$$(e^{At})^{-1} = e^{-At}.$$

Cayley-Hamilton

Notation: For a given polynomial

$$p(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

and an $n \times n$ matrix A, we define

$$p(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I_{n\times n},$$

which is also an $n \times n$ matrix.

Def (Characteristic polynomial of an $n \times n$ matrix A):

$$\Delta(s) := \det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n.$$

Theorem (Cayley-Hamilton). For every $n \times n$ matrix A,

$$\Delta(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I_{n\times n} = 0_{n\times n}.$$

Corollary of the Cayley-Hamilton Theorem: For any given $n \times n$ matrix, for any $k \geqslant 0$, A^k can be written as linear combination of $\{A^{n-1}, A^{n-2}, \cdots, A, I_{n\times n}\}$.

Properties of the transition matrix of and LTI system

Corollary of the Cayley-Hamilton Theorem: For any given $n \times n$ matrix, for any $k \geqslant 0$, A^k can be written as linear combination of $\{A^{n-1}, A^{n-2}, \cdots, A, I_{n\times n}\}$.

- P5. For evert $n \times n$ matrix A, there exist n functions $\alpha_0(t), \alpha_1(t), \cdots, \alpha_{n-1}(t)$ for which

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \quad \forall t \in \mathbb{R}.$$

- P6. For every $n \times n$ matrix A,

$$Ae^{At} = e^{At}A, \quad \forall t \in \mathbb{R}.$$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A, \quad \forall t \in \mathbb{R}.$$

Properties of the transition matrix of and LTI system

- P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geqslant t_0.$$

  - LTI: The function e^{At} is the unique solution of

$$\frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A0} = I, \quad t \geqslant 0.$$

- P.2 For
  - LT
- P3. For
  - LT

$$\begin{aligned}
\mathcal{L}[\frac{d}{dt}e^{At}] &= \mathcal{L}[Ae^{At}] \Leftrightarrow \\
s\mathcal{L}[e^{At}] - e^{A0} &= A\mathcal{L}[e^{At}] \Leftrightarrow \\
s\mathcal{L}[e^{At}] - I &= A\mathcal{L}[e^{At}] \Leftrightarrow \\
(sI - A)\mathcal{L}[e^{At}] &= I \Leftrightarrow \\
\mathcal{L}[e^{At}] = (sI - A)^{-1} &\Leftrightarrow e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]
\end{aligned}$$

- P4. For evert t, τ, $\phi(t, \tau)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

  - LTI: For every $t \in \mathbb{R}$, the function e^{At} is nonsingular and

$$(e^{At})^{-1} = e^{-At}.$$

Some note on computing $\mathcal{L}^{-1}[(sI-A)^{-1}]$

$$e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$$

$$(sI-A)^{-1} = \frac{\begin{bmatrix}\text{polynomials with order} \\ \text{of at most } n-1\end{bmatrix}}{\det(sI-A)} = \frac{\begin{bmatrix}\hat{n}_{1,1}(s) & \cdots & \hat{n}_{1,n}(s) \\ \vdots & \ddots & \vdots \\ \hat{n}_{n,1}(s) & \cdots & \hat{n}_{n,n}(s)\end{bmatrix}}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}}$$

λ_i an eigenvalue of A with multiplicity of m_i, $i \in \{1,\dots,k\}$

(notice $m_1 + m_2 + \cdots + m_k = n$)

Every entry of $(s\mathbf{I}-\mathbf{A})^{-1}$

$$\frac{\hat{n}_{i,j}(s)}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}}$$

$$= \frac{a_{1,1}}{(s-\lambda_1)} + \cdots + \frac{a_{1,m_1}}{(s-\lambda_1)^{m_1}} + \cdots + \frac{a_{k,1}}{(s-\lambda_k)} + \cdots + \frac{a_{k,m_k}}{(s-\lambda_k)^{m_k}}$$

Use

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{1}{n!}t^n e^{-at}, \quad t \geqslant 0$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = e^{at}\sin(bt)$$

$$\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos(bt)$$

$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$: example

Compute e^{At} for $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\{0, -1, 2\}$.

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ -1 & s & -2 \\ 0 & -1 & s-1 \end{bmatrix}, \quad \det(sI - A) = (s-0)(s+1)(s-2) = s(s+1)(s-2)$$

$$(sI-A)^{-1} = \frac{1}{s(s+1)(s-2)} \begin{bmatrix} s^2 - s - 2 & 0 & 0 \\ s-1 & s^2 - s & 2s \\ 1 & s & s^2 \end{bmatrix}$$

$$= \frac{1}{s(s+1)(s-2)} \begin{bmatrix} (s-2)(s+1) & 0 & 0 \\ (s-1) & s(s-1) & 2s \\ 1 & s & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ \frac{s-1}{s(s+1)(s-2)} & \frac{s-1}{(s+1)(s-2)} & \frac{2}{(s+1)(s-2)} \\ \frac{1}{s(s+1)(s-2)} & \frac{1}{(s+1)(s-2)} & \frac{s}{(s+1)(s-2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ \frac{\frac{1}{2}}{s} - \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{6}}{s-2} & \frac{\frac{2}{3}}{(s+1)} + \frac{\frac{1}{3}}{(s-2)} & \frac{\frac{-2}{3}}{(s+1)} + \frac{\frac{2}{3}}{(s-2)} \\ \frac{-\frac{1}{2}}{s} + \frac{\frac{1}{3}}{s+1} + \frac{\frac{1}{6}}{s-2} & \frac{\frac{-1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} & \frac{\frac{1}{3}}{s+1} + \frac{\frac{2}{3}}{s-2} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} - \frac{2}{3}e^{-t} + \frac{1}{6}e^{2t} & \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} & -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} \\ -\frac{1}{2} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \end{bmatrix}$$

For mor examples check out
https://tinyurl.com/wfc75cbm

A brief review of some relevant concept and theory from linear algebra

Eigenvalues and eigenvectors of a matrix

Consider a matrix $A \in \mathbb{C}^{n\times n}$,

$$Ap = \lambda p,$$

- $\lambda \in \mathbb{C}$ is eigenvalue iff we have $p \in \mathbb{C}^{n\times 1}$, $p \neq 0_{n\times 1}$
- Compute λ: $\Delta(A) = \det(\lambda I - A) = 0$; has n roots $\Rightarrow$ n eigenvalues
- Computing eigenvectors: $q \neq 0$ such that $(\lambda I - A)p = 0$, i.e., q is in the nullspace of $(\lambda I - A)$,
- Some of the properties of the eigenvectors
  - When all the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a $n \times n$ matrix A are distinct (multiplicity of all eigenvalues is 1), the nullity of $(\lambda_i I - A)$ is equal to 1. Moreover, the corresponding eigenvector set $\{p_1, \cdots, p_n\}$ is linearly independent.
  - When $\bar{\lambda}$ is an eigenvalue of A with multiplicity of $m \in [2, n]$, then we have $1 \leqslant \text{nullity}(\bar{\lambda} I - A) \leqslant m$.

Basic definitions from linear algebra

- The set of vectors $\{x_1, x_2, \cdots, x_m\}$ in $\mathbb{R}^n$ is said to be **linearly dependent** if and only if there exists real number $\alpha_1, \cdots, \alpha_m$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = 0 \quad (1)$$

- If the only set of α_i's for which (1) holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ then the set of vectors $\{x_1, x_2, \cdots, x_m\}$ is said to be **linearly independent**.

- The **dimension** of a linear space can be defined as the maximum number of linearly independent vectors in the space. Therefore, in $\mathbb{R}^n$, we can find at most n linearly independent vectors.

- **Basis and representation** A set of linearly independent vectors in $\mathbb{R}^n$ is called a basis if every vector in $\mathbb{R}^n$ can be expressed as a unique linear combination of the set. In $\mathbb{R}^n$, any set of n linearly independent vectors can be used as a basis.

- **Rank of a matrix** is the number of linearly independent columns of a matrix.

Solution of $Ax = y$

Theorem: $Ax = y$, $A \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^{n\times 1}$ are given matrices

- A is nonsingular (inverse of A exisits)
  - for every y, $x = A^{-1}y$ is the unique solution
  - for $y = 0_{n\times 1}$, $x = 0_{n\times 1}$ is the unique solution
- $Ax = 0$, $x \neq 0$ if and only if A is singular
  - nullity of A: number of linearly independent solutions of $Ax = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 5 \\ -1 & -2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 5 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

Nullity: 1

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$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -1 & -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nullity: 2

References

[1] Joao P. Hespanha, ``Linear systems theory”, Princeton University Press (Chapter 6)