Linear Systems I
Lecture 4

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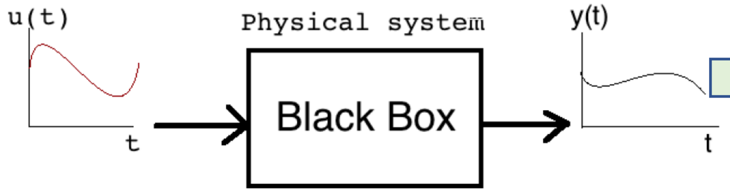
Summary of previous lecture and today's o`utline

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$



Theorem: Let y be an output corresponding to a given input u of a linear system. All outputs corresponding to u can be obtained by

Response=zero-input response + zero-state response

$$y = y_{zs} + y_{zi}$$

To construct all the outputs due to u:

- Find one particular output corresponding to the input u and zero initial condition.
- Final all outputs corresponding to the zero input.

Input-output description (Impulse response)

relaxed and linear : $y(t) = \int_{t_0}^{t} G(t,\tau)\, u(\tau) d\,\tau\,,\ \forall t \geq 0$

relaxed and linear time-invariant: $y(t) = \int_{t_0}^{t} G(t)\, u(t-\tau) d\,\tau\,,\ \forall t \geq 0$

Input-output description in Laplace domain

$$\hat{y}(s) = \hat{G}(s)\,\hat{u}(s), \qquad \forall t \geq 0$$

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

$$y(t) = \psi(t)x(0) + \int_0^t G(t-\tau)u(\tau)d\tau,$$
$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)], \quad \psi(t) = \mathcal{L}^{-1}[\hat{\psi}(s)].$$

SS ↔ Rational TF

Lecture 4 covers

- Zero-state equivalence
- Algebraically equivalent LTI systems
- Solution of LTV systems
  - Solution to Homogeneous Linear systems
  - Transition matrix and its properties

Zero input equivalence

Def(Zero-state equivalence): Two state-space systems are said to be zero-state equivalent if they realize the same transfer function, which means that they exhibit the same forced-response to every input. Zero-state equivalent systems does not necessarily are of the same dimension. The following SS forms are zero-state equivalent.

$$A = \left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right], \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$\bar{A} = \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

Zero state responses of two zero-state equivalent system are the same!

Algebraically equivalent LTI systems

Consider

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

Given T nonsingular, apply change of variable $\bar{x} = Tx$ to write the system in the new state $\bar{x}$

$$\begin{cases} \dot{\bar{x}} = T\dot{x} = T(Ax(t) + Bu(t)) = \underbrace{TAT^{-1}}_{\bar{A}}\bar{x} + \underbrace{TB}_{\bar{B}}u(t) \\ y(t) = C(t)x(t) + D(t)u(t) = \underbrace{CT^{-1}}_{\bar{C}}\bar{x} + \underbrace{D}_{\bar{D}}u(t) \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

Def(Algebraically equivalent) Two continuous-time LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad \text{and} \quad \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

are called algebraically equivalent if and only if there exists a nonsingular T s. t. ($\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$). The corresponding map $\bar{x} = Tx$ is called a similarity transformation or an equivalence transformation.

From rational proper TF to SS: Example (cont'd)

P1. With every input signal u, both systems associate the same set of outputs y. However, the output is generally not the same for the same initial conditions, except for the forced or zero-state response, which is always the same.

P2. the systems are zero-state equivalent, i.e., they have the same transfer function.

$$\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}=C(sI-A)^{-1}B+D$$

$$\begin{aligned}\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}=&CT^{-1}(sI-TAT^{-1})^{-1}TB+D=\\&CT^{-1}(sTT^{-1}-TAT^{-1})^{-1}TB+D=\\&CT^{-1}(T(sI-A)^{-1}T^{-1})TB+D=\\&C(sI-A)^{-1}B+D.\end{aligned}$$

Attention: In general the converse of P2. does not hold, i.e., zero-state equivalence does not imply algebraic equivalence. For two state equations to be equivalent, they must have the same dimension. This is, however, is not required for zero-state equivalent systems.

P3. they have the same eigenvalues.[1]

$$\bar{\Delta}(\lambda)=\det(\lambda I-\bar{A})=\det(\lambda I-A)=\Delta(\lambda)$$

The equivalent state equations have the same characteristic polynomial and consequently the same se of eigenvalues.

$$\begin{aligned}\bar{\Delta}(\lambda)=\det(\lambda I-\bar{A})=&\det(\lambda TT^{-1}-TAT^{-1})=\det(T)\det(\lambda I-A)\det(T^{-1})=\\&=\det(\lambda I-A)\det(T)\det(T^{-1})=\det(\lambda I-A)=\Delta(\lambda).\end{aligned}$$

[1] recall $\det(AB)=\det(A)\det(B)=\det(B)\det(A)$

Solution of an LTV system

We want to study the properties of solutions to SS LTV systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

$A(t) : [0, \infty) \to \mathbb{R}^{n\times n}$, $B(t) : [0, \infty) \to \mathbb{R}^{n\times p}$, $C(t) : [0, \infty) \to \mathbb{R}^{q\times n}$, $D(t) : [0, \infty) \to \mathbb{R}^{q\times p}$. Start by study of

homogeneous linear system: $\dot{x} = A(t)x(t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \; t \geqslant t_0. \quad (1)$

Theorem (Peano-Baker Series). The unique solution to (1) is given by

$$x(t) = \phi(t, t_0)x_0, \qquad (2)$$

$$\phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \cdots$$

- $\phi(t, t_0)$: transition matrix (size $n \times n$)
- The series above is called Peano-Baker series

Properties of the transition matrix of and LTV system

P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geqslant t_0.$$

P.2 For evert fixed t_0, the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geqslant t_0,$$

where e_i is the i^{th} column of identity matrix I_n, or equivalently a column vector of all zero entries except for the i^{th} which is equal to 1.

P3. For every t, s, τ we have

$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau).$$

This property is called the semigroup property.

P4. For evert t, τ, $\phi(t, t_0)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

Properties of the transition matrix of and LTV system

P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geqslant t_0.$$

$$\frac{d}{dt}\phi(t,t_0) = A(t) + A(t)\int_{t_0}^{t} A(\tau_1)d\tau_1 + A(t)\int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 + \cdots = A(t)\phi(t,t_0)$$

$$\phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \cdots$$

Recall

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)d\tau = f\big(t, b(t)\big)\dot{b}(t) - f\big(t, a(t)\big)\dot{a}(t) + \int_{a(t)}^{b(t)} \frac{d}{dt} f(t,\tau)d\tau$$

Proving that the series actually converges and that the solution is unique is beyond the scope of this course.

Properties of the transition matrix of and LTV system

P1. For every t_0, $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t,t_0) = A(t)\phi(t,t_0), \quad \phi(t_0,t_0) = I, \quad t \geqslant t_0.$$

P.2 For evert fixed t_0, the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geqslant t_0,$$

where e_i is the i^{th} column of identity matrix I_n, or equivalently a column vector of all zero entries except for the i^{th} which is equal to 1.

Example:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x \rightarrow \begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = t\, x_1 \end{cases} \rightarrow \begin{cases} x_1(t) = c_1 \\ x_{2(t)} = c_1 \dfrac{t^2}{2} + c_2 \end{cases} \rightarrow \begin{cases} x(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -\dfrac{t_0^2}{2} \end{cases} \rightarrow \quad x(t) = \begin{bmatrix} 1 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} \end{bmatrix} \\ x(t_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{cases} c_1 = 0 \\ c_2 = 1 \end{cases} \rightarrow \quad x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\rightarrow \phi(t, t_0) = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & 1 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & 1 \end{bmatrix} x(t_0)$$

Properties of the transition matrix of and LTV system

P.2 For evert fixed t_0, the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geqslant t_0,$$

where e_i is the i^{th} column of identity matrix I_n, or equivalently a column vector of all zero entries except for the i^{th} which is equal to 1.

Fundamental Matrix of $\dot{x} = A(t)x(t)$:

- Consider a set of n initial condition $x_i(t_0)$, $i \in \{1, \cdots, n\}$.
- For every $x_i(t_0)$ there exists a unique solution $x_i(t)$.
- Arrange these n solutions as $X(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]$
- Note that $\dot{X}(t) = A(t)X(t)$

If n initial condition $x_i(t_0)$, $i \in \{1, \cdots, n\}$ are linearly independent ($X(t_0)$ is non-singular) then $X(t)$ is called a fundamental matrix of $\dot{x} = A(t)x(t)$

- Fundamental matrix $X(t)$ is not unique.
- $\phi(t, t_0)$ is a unique special case of a fundamental matrix.

Let $X(t)$ be any fundamental matrix of $\dot{x} = A(t)x(t)$. Then

$$\phi(t, t_0) = X(t)X^{-1}(t_0).$$

Note that because $X(t)$ is non-singular for all $t \geq t_0$, its inverse is well-defined.

Properties of the transition matrix of and LTV system

Example:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x \rightarrow \begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = t\, x_1 \end{cases} \rightarrow \begin{cases} x_1(t) = c_1 \\ x_{2(t)} = c_1 \dfrac{t^2}{2} + c_2 \end{cases} \rightarrow \begin{cases} x(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -\dfrac{t_0^2}{2} \end{cases} \rightarrow \quad x(t) = \begin{bmatrix} 1 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} \end{bmatrix} \\ x(t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -\dfrac{t_0^2}{2} + 2 \end{cases} \rightarrow \quad x(t) = \begin{bmatrix} 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} + 2 \end{bmatrix} \end{cases}$$

$$\rightarrow \phi(t, t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & \dfrac{t^2}{2} - \dfrac{t_0^2}{2} + 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & \dfrac{t^2}{2} - \dfrac{t_0^2}{2} + 2 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{-1}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & 1 \end{bmatrix} \rightarrow$$

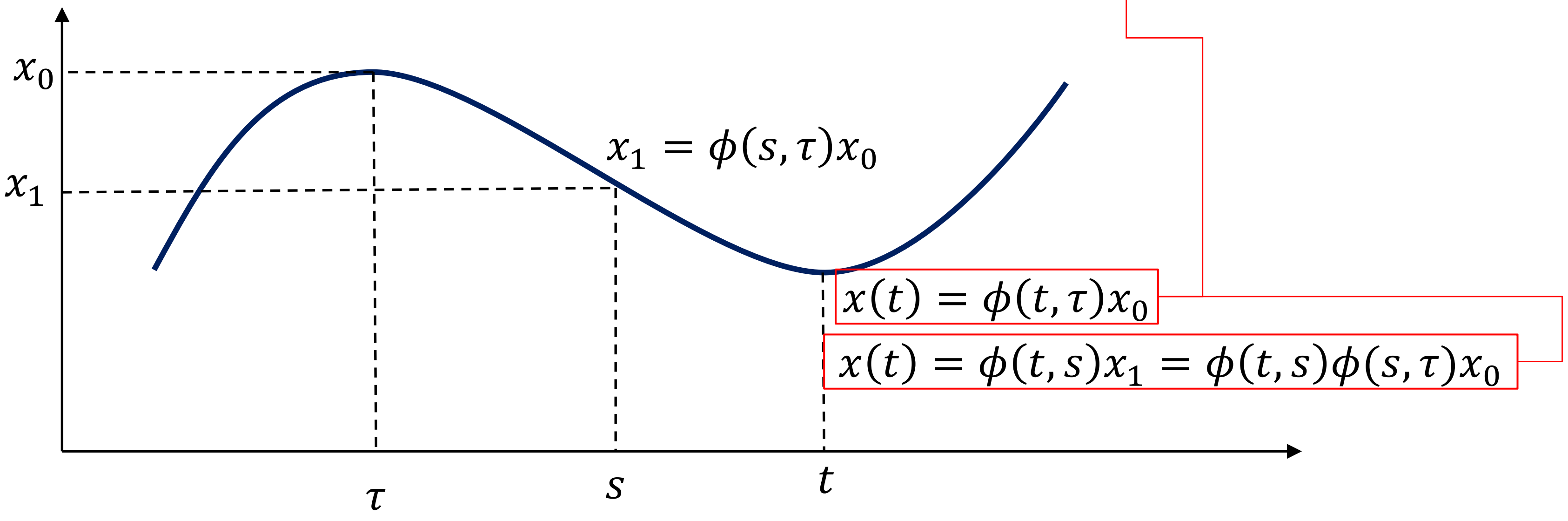
$$x(t) = \begin{bmatrix} 1 & 0 \\ \dfrac{t^2}{2} - \dfrac{t_0^2}{2} & 1 \end{bmatrix} x(t_0)$$

Properties of the transition matrix of and LTV system

P3. For every t, s, τ we have

$$\phi(t,s)\phi(s,\tau) = \phi(t,\tau).$$

This property is called the semigroup property.



P4. For evert t, τ, $\phi(t, t_0)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

From P3 we have $\phi(t, \tau)\phi(\tau, t) = \phi(t, t)$ which gives $\phi(t, \tau)\phi(\tau, t) = I$. From P3 we can also write $\phi(\tau, t)\phi(t, \tau) = \phi(\tau, \tau)$ which gives $\phi(\tau, t)\phi(t, \tau) = I$. Therefore we have $\phi(t, \tau)\phi(\tau, t) = \phi(\tau, t)\phi(t, \tau) = I$. This completes the proof (recall the definition of an inverse of a matrix).

Note: Here, we used $\phi(t, t) = I$ for all t.

Solution of a LTV system

We want to study the properties of solutions to SS LTV systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

$A(t) : [0, \infty) \to \mathbb{R}^{n\times n}$, $B(t) : [0, \infty) \to \mathbb{R}^{n\times p}$, $C(t) : [0, \infty) \to \mathbb{R}^{q\times n}$, $D(t) : [0, \infty) \to \mathbb{R}^{q\times p}$.

Theorem (Variation of constants): The unique solution to LTV SS equation above is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t),$$

where $\phi(t, t_0)$ is the state transition matrix (as defined before).

$$y(t) = \underbrace{C(t)\phi(t, t_0)x_0}_{\text{homogeneous response}} + \underbrace{\int_{t_0}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\text{forced response}}.$$

References

[1] Joao P. Hespanha, ``Linear systems theory”, Princeton University Press (Chapter 4, Chapter 5)