Linear Systems I
Lecture 2

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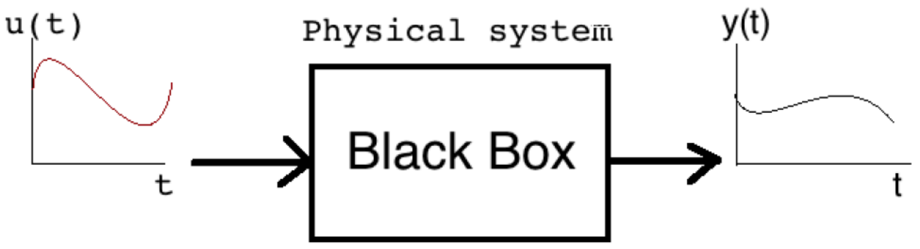
Outline

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$



Assumption: If an excitation or input is applied to the input terminal a unique response or output signal can be measured at the output terminal.

Objective: Discuss some of the basic properties of state-space linear systems

- Basic properties of LTV/LTI systems
  - Causality
  - Linearity
  - Time invariance
- Characterization of all the outputs to a given input
  - Impulse response
  - Laplace transformation (review)
  - Transfer functions

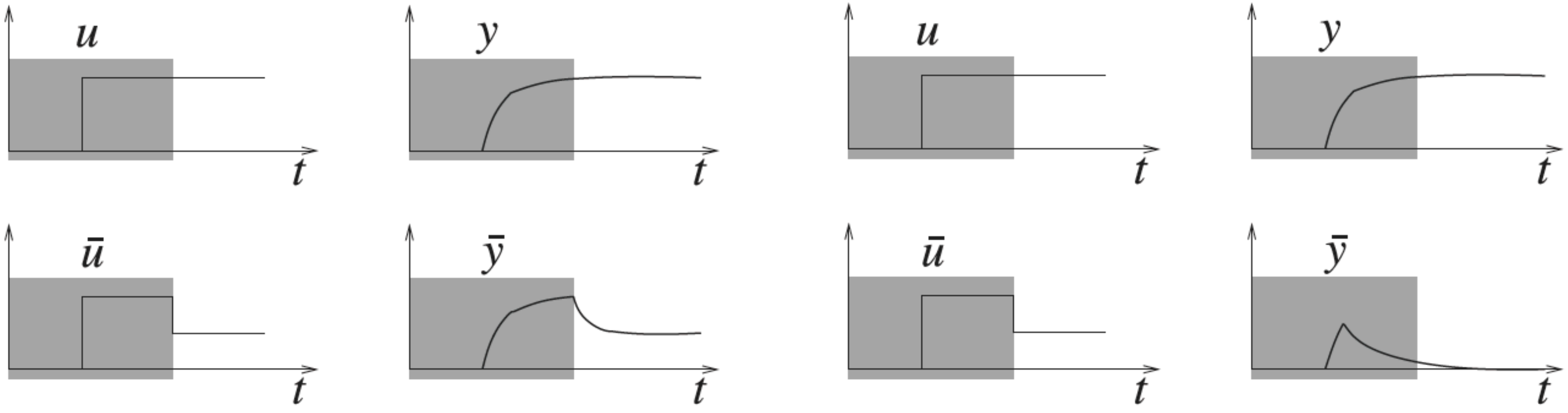
Basic properties of LTV/LTI systems: Causality

Def. A <u>causal</u> or <u>non-anticipatory</u> system is a system whose current output depends on past and current inputs but not on future inputs.

$$\left.\begin{array}{l} x(t_0)=x_0 \\ u(t),\quad t\geqslant t_0 \end{array}\right\} \rightarrow y(t), \qquad \left.\begin{array}{l} x(t_0)=x_0 \\ \bar{u}(t),\quad t\geqslant t_0 \end{array}\right\} \rightarrow \bar{y}(t).$$

Then, if $u(t)=\bar{u}(t)$ for $t\in[t_0,T]$, then $y(t)=\bar{y}(t)$ for $t\in[t_0,T]$.

LTV(LTI) systems are casual!



(a) Causal system

(b) Noncausal system

Figure credit: Ref [1].

Concept of state

Def. The <u>state</u> $x(t_0)$ of a system at time t_0 is the information at t_0 that together with the input $u(t)$, for $t \geqslant t_0$, determine the output $y(t)$ for all $t \geqslant t_0$,

$$\left.\begin{array}{l} x(t_0) \\ u(t), \quad t \geqslant t_0 \end{array}\right\} \to y(t).$$

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

Initial condition
$x(t_0) \in R^n$

Def. A system is said to be lumped if its number of state variables is finite.

$$x(t) = u(t - T),$$
$$y(t) = x(t).$$

Initial condition
$x(\tau) = u(\tau), \quad \tau \in [-T, t_0]$

Def. A system is said to be distributed if it has infinitely many state variables.

Basic properties of LTV/LTI systems: Linearity

Def. A system is linear if and only if for every initial conditions the following hold

$$\left.\begin{array}{l} x(t_0) = x_1 \\ u_1(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \ y_1(t), \qquad \left.\begin{array}{l} x(t_0) = x_2 \\ u_2(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \ y_2(t)$$

we have

$$\left.\begin{array}{l} x(t_0) = \alpha x_1 + \beta x_2 \\ \alpha u_1(t) + \beta u_2(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \ \alpha y_1(t) + \beta y_2(t)$$

LTV(LTI) systems are linear!

or alternatively

$$\left.\begin{array}{l} x(t_0) = x_1 + x_2 \\ u_1(t) + u_2(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \ y_1(t) + y_2(t), \quad \text{(additivity)}$$

$$\left.\begin{array}{l} x(t_0) = \alpha x_1 \\ \alpha u_1(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \ \alpha y_1(t), \quad \text{(homogeneity)}$$

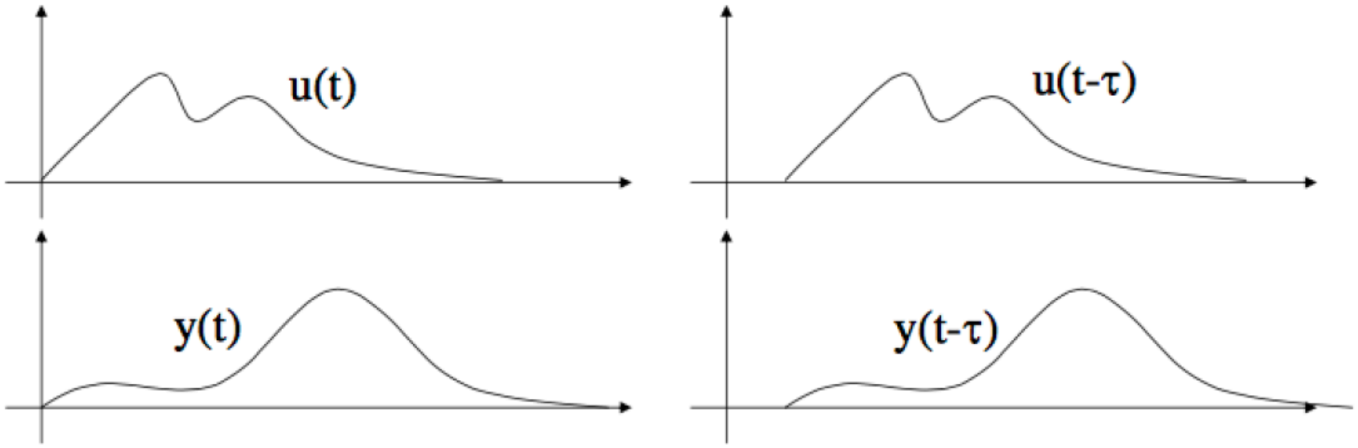
Basic properties of LTV/LTI systems: Time invariance

Def. A system is <u>time invariant</u> if its characteristics do not change with time.

$$\left.\begin{array}{l} x(t_0) = x_0 \\ u(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \; y(t),$$

then the system is time-invariant if and only if

$$\left.\begin{array}{l} x(t_0+\tau) = x_0 \\ \bar{u}(t) = u(t-\tau), \quad t \geqslant t_0+\tau \end{array}\right\} \rightarrow \; \bar{y}(t) = y(t-\tau), \quad t \geqslant t_0+\tau \quad \text{(time shifting)}$$



A time invariance system

Figure credit: Ref [1].

Characterization of all the outputs to a given input

Def If the input $u(t) = 0$ for all $t \geqslant t_0$, then the output will be excited exclusively by the initial state $x(t_0)$. This output is called zero-input response (homogeneous response) and will be denoted by y_h or y_{zi}

$$\left.\begin{array}{l} x(t_0) \\ u(t) = 0, \quad t \geqslant t_0 \end{array}\right\} \rightarrow \; y_{zi}(t).$$

Def If the initial state $x(t_0)$ is zero, the output will be excited exclusively by the input . This output is called zero-state response (forced response) and will be denoted by y_f or y_{zs}

$$\left.\begin{array}{l} x(t_0) = 0 \\ u(t), \quad t \geqslant t_0 \end{array}\right\} \rightarrow \; y_{zs}(t).$$

Input-output description

Theorem: Let y be an output corresponding to a given input u of a linear system. All outputs corresponding to u can be obtained by

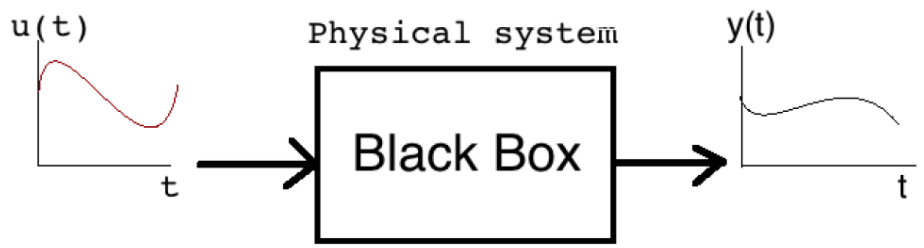
Response=zero-input response + zero-state response

$$y = y_{zs} + y_{zi}$$

To construct all the outputs due to u:

- Find one particular output corresponding to the input u and zero initial condition.
- Final all outputs corresponding to the zero input.

Input-output description: Impulse response



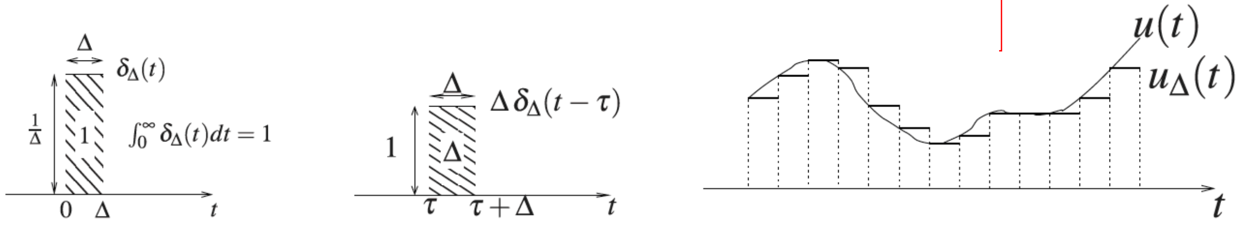
Impulse response : mathematical description of zero-state response.

Assumption: System is relaxed

Def. (relaxed system): A system is said to be relaxed at t_0 if its initial state $x(t_0)$ is 0. In this case the output $y(t), t \geq\ t0$ is excited exclusively by the input $u(t)$ for $, t \geq\ t0$.

Input-output description: Impulse response

Consider a relaxed SISO system



(a) Pulse (b) Time-shifted pulse (c) Step approximation

Step approximation to a continuous-time signal.

Figure credit: Ref [1].

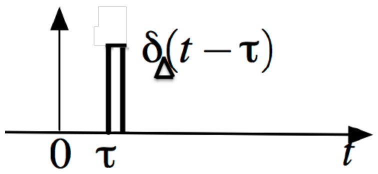
Approximation to an input signal $u: [0,\infty) \to R$:

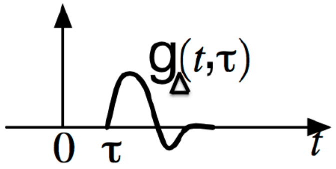
$$u_\Delta(t) = \sum_{k=0}^{\infty} u(k\Delta)\Delta\,\delta_\Delta(t-k\Delta), \qquad \forall t \geq 0$$

Linearity

For each $\tau \geq 0$, let $g_\Delta(t,\tau), t \geq 0$ be an output corresponding to input $\delta_\Delta(t-\tau)$:

$$\delta_\Delta(t-\tau) \longmapsto g_\Delta(t,\tau)$$





Approximation to an input signal $u: [0,\infty) \to R$:

$$u_\Delta(t) \longmapsto y_\Delta(t) = \sum_{k=0}^{\infty} g_\Delta(t,k\Delta)u(k\Delta)\Delta, \qquad \forall t \geq 0$$

When $\Delta \to 0$ we have $u_\Delta(t) \to u(t)$:

$$u(t) \longmapsto y(t) = \lim_{\Delta\to 0} y_\Delta(t) = \lim_{\Delta\to 0} \sum_{k=0}^{\infty} g_\Delta(t-k\Delta)u(k\Delta)\Delta = \int_0^\infty g(t,\tau)\,u(\tau)d\,\tau\,, \quad \forall t \geq 0$$

$$g(t,\tau) = \lim_{\Delta\to 0} g_\Delta(t,k\Delta)$$

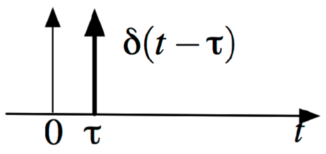
Input-output description: Impulse response

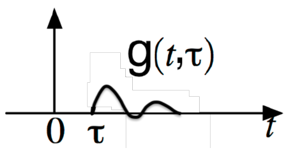
Consider a relaxed SISO system

Zero-state response of a SISO linear system given a u: $[0,\infty) \to R$:

$$u(t) \longmapsto y(t) = \int_{t_0}^{\infty} g(t,\tau)u(\tau)d\,\tau\,, \qquad \forall t \geq 0$$

$$g(t,\tau) = \lim_{\Delta\to 0} g_\Delta(t-k\Delta)$$





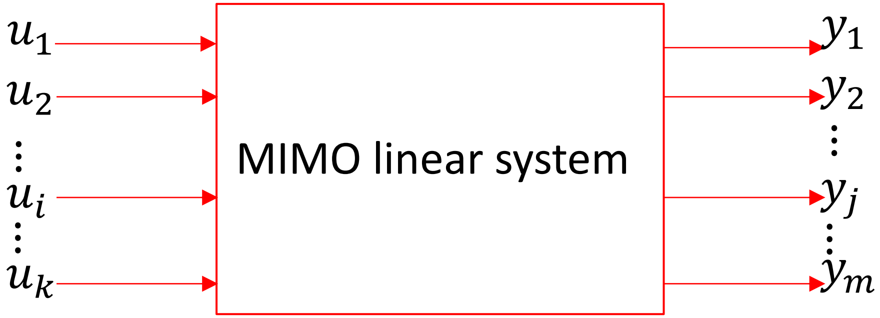
$$u(t) \longmapsto y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\,\tau\,, \qquad \forall t \geq 0$$

We are concerned with **causal** systems:

$$\text{causal} \;\Leftrightarrow\; g(t,\tau) = 0 \quad \text{for} \quad \forall \tau > t$$

i.e., the output is not going to appear before we apply an input.

Input-output description: Impulse response of MIMO systems



Impulse response for MIMO : mathematical description of zero-state response for a system with k input and m output (**linear**, **causal** and **relaxed** system)

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

where

$$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1k}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \cdots & g_{2k}(t,\tau) \\ \vdots & \vdots & & \vdots \\ g_{m1}(t,\tau) & g_{m2}(t,\tau) & \cdots & g_{mk}(t,\tau) \end{bmatrix}$$

$g_{ij}(t,\tau)$ is the system's output at time t at the i^{th} output due to an impulse at time τ at the j^{th} input terminal, while the input of there terminals being identically zero.

Input-output description: Impulse response of LTI systems

Zero-state response of a SISO linear system given a $u: [0, \infty) \to R$:

$$u(t) \longmapsto y(t) = \int_{t_0}^{t} u(\tau) g(t, \tau)\, d\,\tau, \qquad \forall t \geq 0$$

Zero-state response of a SISO LTI given a $u: [0, \infty) \to R$:

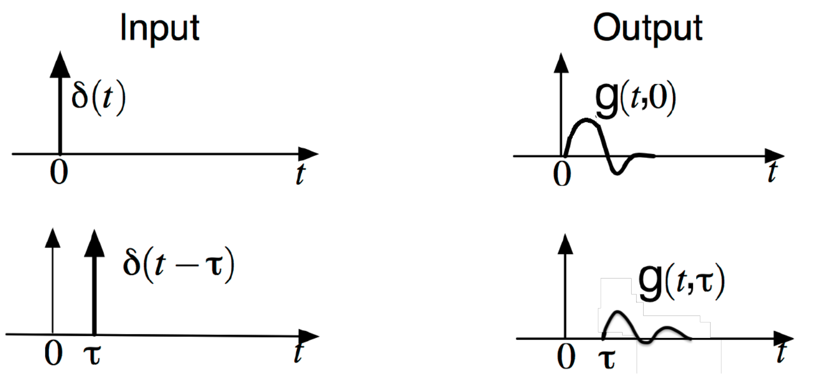
$$u(t) \longmapsto y(t) = \int_{t_0}^{t} g(t - \tau)\, u(\tau)\, d\,\tau, \qquad \forall t \geq 0$$

Or equivalently

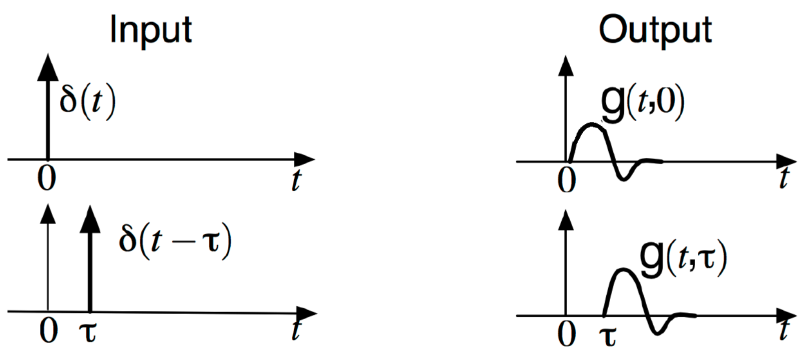
$$u(t) \longmapsto y(t) = \int_{t_0}^{t} g(t)\, u(t - \tau)\, d\,\tau, \qquad \forall t \geq 0$$

Time-invariant system: $g(t, \tau) = g(t - \tau, \tau - \tau) = g(t - \tau, 0) = g(t - \tau)$

$g(t) = \mathrm{g}(\mathrm{t} - 0)$: The output at time t due to an impulse input applied at time 0.



LTV



LTI

Input-output description: Impulse response

Input-output description (Impulse response) for relaxed and linear system

$$y(t) = \int_{t_0}^{t} G(t,\tau)\, u(\tau) d\,\tau\,, \qquad \forall t \geq 0$$

$G(t,\tau)$ is the system's output at time t due to an impulse at time τ.

Input-output description (Impulse response) for relaxed and linear time-invariant system

$$y(t) = \int_{t_0}^{t} G(t)\, u(t-\tau) d\,\tau\,, \qquad \forall t \geq 0$$

$G(t)$ is the system's output at time t due to an impulse at time 0.

Laplace Transform (review)

Def. Given a Continuous-time signal $x(t)$, $t \in \mathbb{R}_{\geqslant 0}$, its (unilateral) Laplace transform is given by

$$\mathcal{L}[x(t)] = \hat{x}(s) = \int_0^\infty x(t)e^{-st}\,dt, \quad s \in \mathbb{C}.$$

Some properties of Laplace transform:

- $\mathcal{L}[\dot{x}(t)] = s\hat{x}(s) - x(0), \quad s \in \mathbb{C}.$
- $\mathcal{L}[(x \star y)(t)] = \mathcal{L}\left[\int_0^t x(\tau)y(t-\tau)d\tau\right] = \hat{x}(s)\,\hat{y}(s), \quad s \in \mathbb{C}.$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

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Transfer function of a LTI system

Every linear, time-invariant system has a transfer function.

$$\hat{G}(s) = \mathcal{L}\big[G(t)\big] = \int_0^{\infty} G(t)e^{-st}\,dt, \quad s \in \mathbb{C},$$

$$\mathcal{L}[(x \star y)(t)] = \mathcal{L}\big[\textstyle\int_0^t x(\tau)y(t-\tau)d\tau\big] = \hat{x}(s)\,\hat{y}(s), \quad s \in \mathbb{C}.$$

Input-output description for relaxed and linear time-invariant system

$$y(t) = \int_{t_0}^{t} G(t)\,u(t-\tau)d\,\tau, \qquad \forall t \geq 0$$

Input-output description in Laplace domain

$$\hat{y}(s) = \hat{G}(s)\,\hat{u}(s), \qquad \forall t \geq 0$$

References

[1] Joao P. Hespanha, ``Linear systems theory”, Princeton University Press (Chapter 3)