Linear Systems I
Lecture 1

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What?

Why?

How?

Objective of this course

Study linear ordinary differential equations of the form below

$$\text{state equation:} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad (1)$$
$$\text{output equation:} \quad y(t) = C(t)x(t) + D(t)u(t), \qquad (2)$$

- $\dot{x}(t) = dx(t)/dt$ denotes the derivative of $x(t)$ w.r.t time
- $x(t) : [0, \infty) \to \mathbb{R}^n$: the system state
- $u(t) : [0, \infty) \to \mathbb{R}^k$: the system inputs
- $y(t) : [0, \infty) \to \mathbb{R}^m$: the system outputs
- $A(t) : [0, \infty) \to \mathbb{R}^{n\times n}$, $B(t) : [0, \infty) \to \mathbb{R}^{n\times k}$, $C(t) : [0, \infty) \to \mathbb{R}^{m\times n}$, and $D(t) : [0, \infty) \to \mathbb{R}^{m\times k}$ are matrices of appropriate dimensions

State Space Model

Linear time-varying system, or for short LTV system

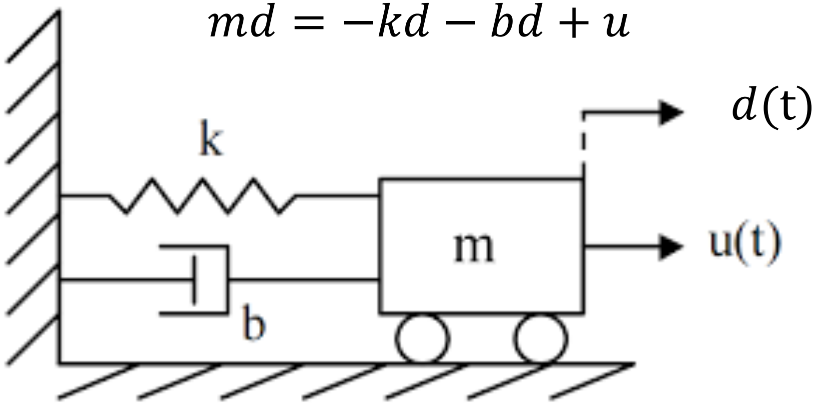
LTI systems: *linear time invariant* systems

$$\begin{aligned}\dot{x}(t) =& Ax(t) + Bu(t), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\\ y(t) =& Cx(t) + Du(t), \quad y \in \mathbb{R}^m.\end{aligned}$$

Why do we study linear state space systems?

- LTV systems are useful in many application areas.
  - Models of mechanical systems (force versus velocity laws for friction; force versus displacement laws for springs) or electrical systems (linear voltage versus current laws for resistors) whose parameters (for example, the stiffness of a spring or the inductance of a coil) change in time.

$$m\ddot{d} = -kd - b\dot{d} + u$$



State-space representation: describing the system equations with a set of first order differential equations

$$x_1 = d \rightarrow \dot{x}_1 = \dot{d} = x_2$$

$$x_2 = \dot{d} \rightarrow \dot{x}_2 = \ddot{d} = -\frac{k}{m}d - \frac{b}{m}\dot{d} + \frac{1}{m}u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}}_{B}\mathrm{u}$$

Let say we can only measure the displacement of the mass

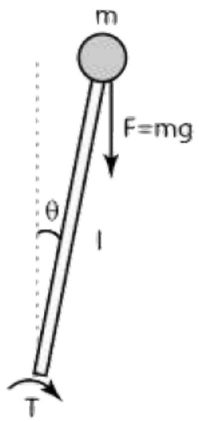
$$y = d = \underbrace{[1 \quad 0]}_{C}\, x + \underbrace{0}_{D}\, u$$

Linear time-invariant (LTI) system: $\begin{cases}\dot{x} = Ax + Bu\\ y = Cx + Du\end{cases}$

Why do we study linear state space systems?

- But linear laws are only approximations to more complex nonlinear relations!
  - More reasonable class of systems to study appears to be

$$\dot{x}(t) = f(x(t), u(t)), \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k,$$
$$y(t) = g(x(t), u(t)), \quad y \in \mathbb{R}^m.$$

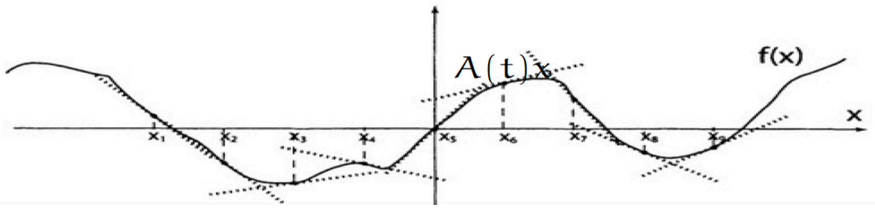


From Newtons law: $ml^2\ddot{\theta} = mgl\sin(\theta) - b\dot{\theta} + T$
Let $ml = 1$, $\frac{b}{ml} = 1$.

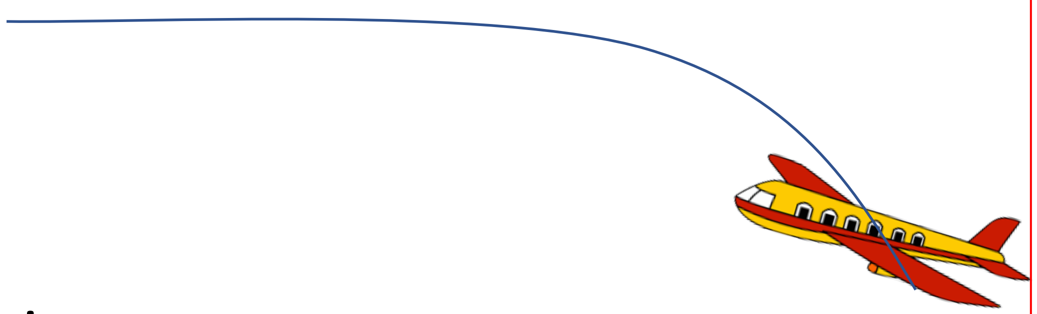
$$\begin{cases} x_1 = \theta, \\ x_2 = \dot{\theta} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = T + g\sin(x_1) - x_2 \\ y = x_1 \end{cases}$$

Why do we study linear state space systems?

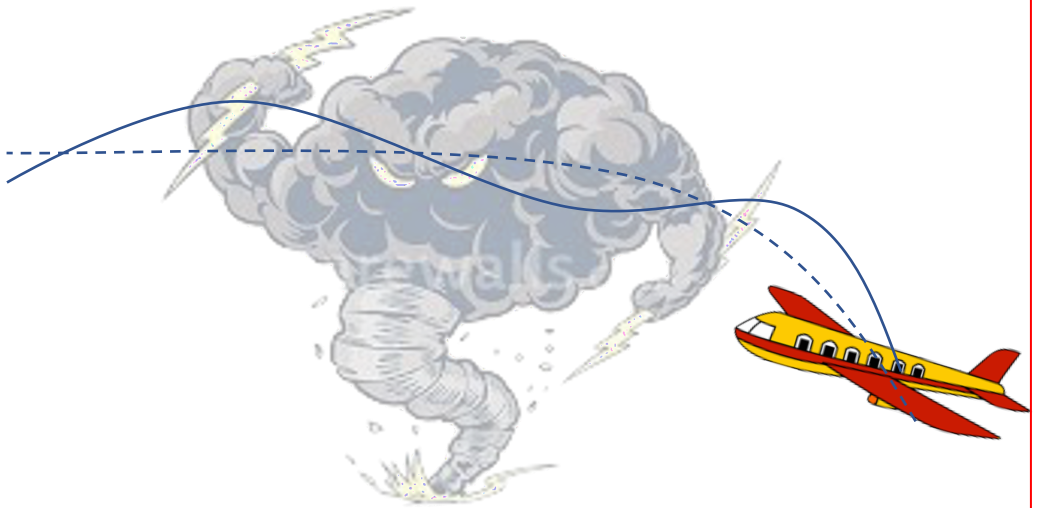
- LTV systems: linearizing a non-linear system around a trajectory
  - One often uses the full nonlinear dynamics to design an optimal trajectory to guide the system from its initial state to a desired final state.
  - One needs to ensure that the system will actually track this trajectory in the presence of disturbances.
  - One solution is to linearize the nonlinear system (i.e. approximate it by a linear system) around the optimal trajectory;
  - the approximation is accurate as long as the nonlinear system does not drift too far away from the optimal trajectory.
  - The result of the linearization is a LTV system, which can be controlled using the methods developed in this course.
  - If the control design is done well, the state of the nonlinear system will always stay close to the optimal trajectory, hence ensuring that the linear approximation remains valid.



Nominal optimal trajectory:



In practice:



Linearization about a nominal solution

- LTV systems: linearizing a non-linear system around a trajectory

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\\ y(t) &= g(x(t), u(t)), \quad y \in \mathbb{R}^m.\end{aligned}$$

Let $x^{\text{sol}} : [0, \infty)$, $u^{\text{sol}} : [0, \infty)$, $y^{\text{sol}} : [0, \infty)$ be a nominal trajectory, i.e.,

$$\begin{aligned}\dot{x}^{\text{sol}}(t) &= f(x^{\text{sol}}(t), u^{\text{sol}}(t)), \quad x^{\text{sol}}(0) = x_0^{\text{sol}},\\ y^{\text{sol}}(t) &= g(x^{\text{sol}}(t), u^{\text{sol}}(t)),\end{aligned}$$

Small perturbation around nominal trajectory for all $t \geqslant 0$ $(t \in \mathbb{R}_{\geqslant 0})$

- perturbation in control input: $u(t) = u^{\text{sol}}(t) + \delta u(t)$
- perturbation in initial conditions: $x(0) = x_0^{\text{sol}} + \delta x(0)$

Results in small perturbation in

$$x(t) = x^{\text{sol}}(t) + \delta x(t), \quad y(t) = y^{\text{sol}}(t) + \delta y(t)$$

Linearization about a nominal solution

To investigate how $x(t)$ and $y(t)$ are perturbed, we are interested in dynamics of $\delta\dot{x}(t)$:

$$\delta x(t) = x(t) - x^{sol}(t), \quad \delta y(t) = y(t) - y^{sol}(t)$$

$$\delta\dot{x}(t) = \dot{x}(t) - \dot{x}^{sol} = f(x(t), u(t)) - f(x^{sol}(t), u^{sol}(t)) =$$

$$f(x^{sol}(t) + \delta x(t), u^{sol}(t) + \delta u(t)) - f(x^{sol}, u^{sol}) =$$

$$f(x^{sol}, u^{sol}) + \frac{\partial f}{\partial x}(x^{sol}, u^{sol})\delta x(t) + \frac{\partial f}{\partial u}(x^{sol}, u^{sol})\delta u(t) + o(\|\delta x(t)\|^2, \|\delta u(t)\|^2) - f(x^{sol}, u^{sol})$$

For small perturbations, then we obtain

$$\boxed{\delta\dot{x}(t) = \frac{\partial f}{\partial x}(x^{sol}, u^{sol})\delta x(t) + \frac{\partial f}{\partial u}(x^{sol}, u^{sol})\delta u(t)}$$

In a similar way obtain the following for the output

$$\boxed{\delta y(t) = \frac{\partial g}{\partial x}(x^{sol}, u^{sol})\delta x(t) + \frac{\partial g}{\partial u}(x^{sol}, u^{sol})\delta u(t)}$$

Linearization about a nominal solution

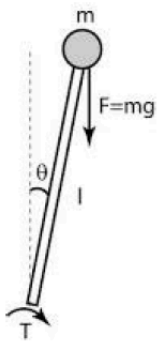
Local linearization of the original nonlinear model around nominal trajectory (x^{sol}, u^{sol})

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t),$$
$$\delta y(t) = C(t)\delta x(t) + D(t)\delta u(t)$$

where

$$A(t) = \frac{\partial f}{\partial x}(x^{sol}(t), u^{sol}(t)),$$
$$B(t) = \frac{\partial f}{\partial u}(x^{sol}(t), u^{sol}(t)),$$
$$C(t) = \frac{\partial g}{\partial x}(x^{sol}(t), u^{sol}(t)),$$
$$D(t) = \frac{\partial g}{\partial u}(x^{sol}(t), u^{sol}(t)),$$

Linearization about a nominal solution: example



From Newtons law: $ml^2\ddot{\theta} = mgl\sin(\theta) - b\dot{\theta} + T$
Let $ml = 1$, $\frac{b}{ml} = 1$. Linearize this system around constant angular velocity $\dot{\theta} = \omega$ trajectory, started at $\theta(0) = 0$. The output of the system we monitor is the angle of rotation.

$$\begin{cases} x_1 = \theta, \\ x_2 = \dot{\theta} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = T + g\sin(x_1) - x_2 \\ y = x_1 \end{cases}$$

- Constant angular velocity trajectory: $\begin{cases} x_1^{sol}(t) = \omega t + x_1(0) = \omega t \\ x_2^{sol}(t) = \omega, \end{cases}$ $\begin{cases} x_1^{sol}(0) = 0, \\ x_2^{sol}(0) = \omega. \end{cases}$
- $(x_1^{sol}(t), x_2^{sol}(t))$ should satisfy the equations of the motion of the pendulum:

$$\begin{cases} \dot{x}_1^{sol} = x_2^{sol} \\ \dot{x}_2^{sol} = T^{sol} + g\sin(x_1^{sol}) - x_2^{sol} \end{cases} \Rightarrow \begin{cases} \omega = \omega \\ 0 = T^{sol} + g\sin(\omega t) - \omega \end{cases} \Rightarrow T^{sol} = -g\sin(\omega t) + \omega.$$

- Linearized model is

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t),$$
$$\delta y(t) = C(t)\delta x(t) + D(t)\delta u(t)$$

where

$$A(t) = \frac{\partial f}{\partial x}(x^{\mathbf{sol}}(t), u^{\mathbf{sol}}(t)) = \begin{bmatrix} 0 & 1 \\ g\cos(\omega t) & -1 \end{bmatrix}, \quad B(t) = \frac{\partial f}{\partial u}(x^{\mathbf{sol}}(t), u^{\mathbf{sol}}(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C(t) = \frac{\partial g}{\partial x}(x^{\mathbf{sol}}(t), u^{\mathbf{sol}}(t)), = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D(t) = \frac{\partial g}{\partial u}(x^{\mathbf{sol}}(t), u^{\mathbf{sol}}(t)) = 0,$$

Linearization about a equilibrium point

- LTI systems: linearizing a non-linear system around an equilibrium point.

Equilibrium point: A pair $(x^{eq}, u^{eq}) \in \mathbb{R}^n \times \mathbb{R}^k$ is called an equilibrium point of

$$\dot{x}(t) = f(x(t), u(t)), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^k, \tag{5a}$$
$$y(t) = g(x(t), u(t)), \quad y \in \mathbb{R}^m. \tag{5b}$$

if $f(x^{eq}, u^{eq}) = 0$. In this case $u(t) = u^{eq}$, $x(t) = x^{eq}$, $y(t) = y^{eq} = g(x^{eq}, u^{eq})$ is a solution to (5).

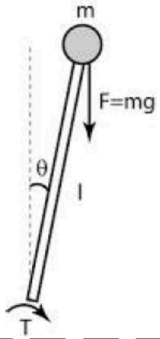
Linearizing around (x^{eq}, u^{eq}) gives the following LTI system

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t),$$
$$\delta y(t) = C\delta x(t) + D\delta u(t)$$

$$A = \frac{\partial f}{\partial x}(x^{eq}, u^{eq}), \quad B = \frac{\partial f}{\partial u}(x^{eq}, u^{eq}),$$
$$C = \frac{\partial g}{\partial x}(x^{eq}, u^{eq}), \quad D = \frac{\partial g}{\partial u}(x^{eq}, u^{eq}),$$

- Feedback linearization Example :

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + G(q) = F, \qquad q \in \mathbb{R}^k,\ F \in \mathbb{R}^k,$$



From Newtons law:

$$ml^2\ddot{\theta} = mgl\sin(\theta) - b\dot{\theta} + T$$

$$F = u = u_{nl}(q,\dot{q}) + M(q)v, \qquad u_{nl}(q,\dot{q}) = B(q,\dot{q})\dot{q} + G(q),$$

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + G(q) = F, \Longrightarrow \ddot{q} = v \Longrightarrow$$

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & I \end{bmatrix} v, \qquad x := \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2k}$$

Feedback linearization

- Feedback linearization: *strict feedback form*

$$\dot{x}_1 = f_1(x_1) + x_2,$$
$$\dot{x}_2 = f_2(x_1, x_2) + u.$$

To feedback linearize, let

$$z_2 := f_1(x_1) + x_2$$

Then

$$\dot{x}_1 = z_2,$$
$$\dot{z}_2 = \frac{\partial f_1}{\partial x_1}(x_1)\dot{x}_1 + \dot{x}_2 = \frac{\partial f_1}{\partial x_1}(x_1)(f_1(x_1) + x_2) + f_2(x_1, x_2) + u.$$

Now, define

$$u = u_{nl}(x_1, x_2) + v, \quad u_{nl}(x_1, x_2) = -\frac{\partial f_1}{\partial x_1}(x_1)(f_1(x_1) + x_2) - f_2(x_1, x_2),$$

Then, we obtain the following LTI system

$$\dot{x}_1 = z_2,$$
$$\dot{z}_2 = v.$$

Linear State Space Systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

Some terminology:

- the system above is called *linear time-varying (LTV)* system
- when u takes scalar values $(k = 1)$: *single input (SI)*; otherwise *multiple input (MI)*
- when y takes scalar values $(m = 1)$: *single output (SO)*; otherwise *multiple output (MO)*
- when there is no state (n=0), i.e, $y(t) = D(t)u(t)$ the system is called *memoryless*.

Study of linear state space systems: questions of interest

Learn something about linear systems:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t), \qquad (\star)$$

Questions of interest:

- Is this system stable?
  - The zero solution $x(t) = 0$ of a zero input LTV system is stable if, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $\|x(0)\| \leqslant \delta, \text{then} \|x(t)\| \leqslant \epsilon$, for all $t \in \mathbb{R}_{\geqslant 0}$.
- Does this system converge? (Asymptotic stability)
  - If in addition to being stable, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, we have $x(t) \to 0$ as $t \to \infty$.
- Is this system controllable?
  - An LTV system is called controllable if and only if for all $x_0 \in \mathbb{R}^n$ and for all $\hat{x} \in \mathbb{R}^n$, and for all finite $T > 0$, there exists $u(t) : [0, T] \to \mathbb{R}^k$ such that the solution of system $(\star)$ with initial condition $x(0) = x_0$ under the input $u(t)$ is such that $x(T) = \hat{x}$.
- Is this system observable?
- $\vdots$

- Unfortunately, answering to our questions from the definition is not tractable (impossible).
  - This would require calculating all trajectories that start at all initial conditions.
  - For example to check for controllability, except for trivial cases (like the linear system $\dot{x}(t) = u(t)$) this calculation is intractable, since the initial states, x_0, the times T of interest, and the possible input trajectories $u(t) : [0, T] \to \mathbb{R}^k$ are all infinite.
- Fortunately, linear algebra can be used to answer the question without even computing a single solution.

Example: Consider a LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^k. \quad (\star)$$

Theorem: An LTI system is asymptotically stable if all eigenvalues of A have negative real parts.

Theorem: An LTI system is controllable iff the matrix $[B \ AB \ \cdots, A^{n-1}B] \in \mathbb{R}^{n \times nk}$ is rank n.

Phase portraits for

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x$$

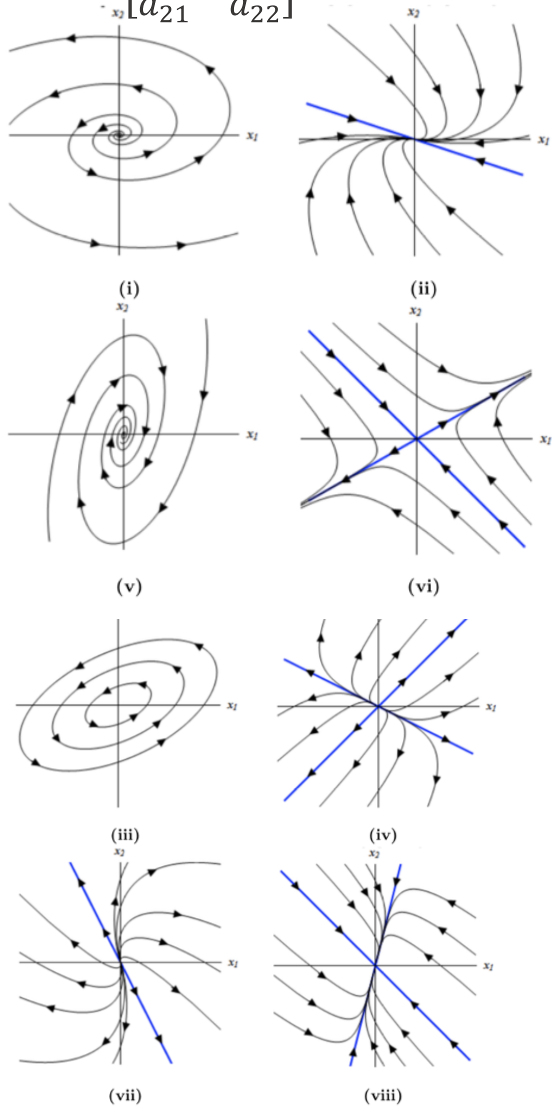


Photo by courtesy of tinyurl.com/fzy48hrc

How do we carry out our study?

Learn something about linear systems: How?

Linear systems theory brings together two areas of mathematics: algebra and analysis.

- As we will soon see, the state space, $\mathbb{R}^n$, of the systems has both an algebraic structure (it is a vector space) and a topological structure (it is a normed space).
- The algebraic structure allows us to perform linear algebra operations, compute projections, eigenvalues, etc.
- The topological structure, on the other hand, forms the basis of analysis, the definition of derivatives, etc.

The **main point of linear systems theory is** to exploit the algebraic structure to develop tractable "algorithms" that allow us to answer analysis questions which appear intractable by themselves.

Our objective: a short summary

Learn something about linear systems:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

exploit the *algebraic* structure to develop tractable "algorithms" that allow us to answer *analysis* questions (stability, convergence, controllability, etc) which appear intractable by themselves