Linear Systems I
Lecture 9

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Reading: Ch 5.3, 5.4 and Example 5.5, Ch 3.9 and Ch. 3.11 of Ref [1].
Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

Outline

- Internal stability of LTV/LTI systems
  - eigenvalue test
  - a note on internal stability of LTV systems
  - Lyapunov method

Stability of LTV systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

Stability addresses what happens to our solutions as time increases

- do they remain bounded
- will they get progressively smaller
- they diverge to infinity

Response is due to : $\underbrace{\text{response due to } x_0}_{\text{internal stability}} + \underbrace{\text{response due to } u}_{\text{Input-output stability}}$

Lets start with Internal stability:

Recall homogeneous system,

$$\dot{x} = A(t)x, x(t_0) = x_0 \in \mathbb{R}^n$$

Our solution is

$$x(t) = \phi(t, t_0)x_0, \quad t \geqslant t_0$$

Lyapunov stability theorem

Consider

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

Theorem The following five conditions are equivalent for the **LTI** system above

1. The system is asymptotically stable
2. The system is exponentially stable
3. All the eigenvalues of A have strictly negative real parts
4. For every $Q > 0$, $\exists$ a unique solution P for the following Lyapunov equation

   $$A^\top P + PA = -Q$$

   Moreover P is symmetric and positive definite.
5. $\exists$ $P > 0$ for which the following Lyapunov matrix inequality holds

   $$A^\top P + PA < 0$$

Stability of LTI systems (Lyapunov method)

$$\dot{x} = Ax, \quad x(t_0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

Proof: Think about how you can prove (2)⇒(4) if one tells you the candidate solution is $P = \int_0^t e^{A^\top t} Q e^{A t} dt$. In your proof you need to show that for every positive definite Q, P is finite, unique positive definite matrix that satisfies $A^\top P + PA = -Q$.

Next we show (4)⇒(2) , i.e.,

$$\exists\ P \succ 0 \text{ and } Q \succ 0 \text{ s.t. } A^\top P + PA = -Q \implies \underbrace{\|x\| \leqslant \kappa \|x(0)\| e^{-ct}}_{(\star)\text{ is exponentially stable}},$$

$x(t)$ arbitrary trajectory of $(\star)$

$$V(t) = x^\top(t) P x(t) \succ 0, \quad \text{where } A^\top P + PA = -Q$$

$$\dot{V}(t) = \dot{x}^\top(t) P x(t) + x^\top(t) P \dot{x}(t) \Rightarrow \dot{V}(t) = x^\top (A^\top P + PA) x = -x^\top Q x \prec 0$$

Then for all $\forall t \geqslant 0$

$$V(t) \leqslant V(0), \implies x^\top(t) P x(t) \leqslant x^\top(0) P x(0) \implies \|x(t)\|^2 \leqslant \frac{x^\top(0) P x(0)}{\lambda_{min}[P]}$$

Starting from any initial condition states stay bounded: the system is stable!

Proof continued from previous page

$$\dot{V}(t) = -x^\top Q x \leqslant -\lambda_{min}[Q]\|x\|^2 \leqslant -\frac{\lambda_{min}[Q]}{\lambda_{max}[P]} V(t), \quad \forall t \geqslant 0.$$

Lemma

(Comparison Lemma) *Let $v(t)$ be a differentiable scaler signal for which*

$$\dot{v}(t) \leqslant \mu v(t), \quad \forall t \geqslant t_0,$$

for some constant $\mu \in \mathbb{R}$. Then

$$v(t) \leqslant v(t_0) e^{\mu(t-t_0)}, \quad \forall t \geqslant t_0.$$

$$V(t) \leqslant V(0) e^{-ct}, \quad \forall t \geqslant 0, \qquad c := \frac{\lambda_{min}[Q]}{\lambda_{max}[P]}$$

$$\|x(t)\|^2 \leqslant \frac{V(0)}{\lambda_{min}[P]} e^{-ct}, \quad \forall t \geqslant 0$$

$$\|x(t)\|^2 \leqslant \frac{\lambda max[P]}{\lambda_{min}[P]} e^{-ct} \|x(0)\|^2, \quad \forall t \geqslant 0$$

$$\|x(t)\| \leqslant \sqrt{\frac{\lambda max[P]}{\lambda_{min}[P]}} e^{-\frac{c}{2}t} \|x(0)\|, \quad \forall t \geqslant 0$$

$\|x(t)\|$ converges to zero exponentially fast, as $t \to \infty$: system is exponentially stable

Consider

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^p$$

where (A, B) is controllable, i.e., we can always find a state feedback gain $K \in \mathbb{R}^{p \times n}$ such that the feedback controller $u = -Kx$ stabilizes the system. That is $A - BK$ is a Hurwitz matrix (all its eigenvalues have negative real part).

$$\dot{x} = Ax + Bu = Ax + B(-Kx) = (A - BK)x.$$

In the following we show how you can synthesize one of those gains using Lyaunov stability results and Matlab LMI tool box.

For a given $K \in \mathbb{R}^{p \times n}$, $A - BK$ is asymptotically stable if and only if

$$\begin{cases} (A - BK)^\top P + P(A - BK) \prec 0, \\ P \succ 0. \end{cases}$$

Multiply these matrix inequalities from let and right by $Q = P^{-1}$, we obtain the equivalent set of equations

$$\begin{cases} Q(A - BK)^\top + (A - BK)Q \prec 0, \\ Q \succ 0. \end{cases}$$

Let $X = KQ$, then we obtain

$$\begin{cases} Q A^\top - X^\top B^\top + A Q - B X \prec 0, \\ Q \succ 0. \end{cases}$$

The equations above are linear matrix inequalities (LMIs) in variables (X, Q). You can use Matlab's LMI solver to obtain a solution (X, Q). Once you have the solution, then your stabilizing feedback gain is

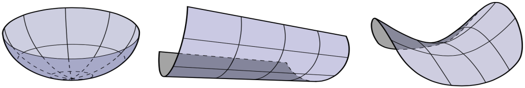
$$K = XQ^{-1}.$$

Def(positive-definite matrix): A symmetric $n \times n$ matrix Q is positive-definite if

$$x^\top Q x > 0, \quad \forall x \in \mathbb{R}^n \backslash 0 \quad (\star)$$

when $>$ is replaced by $<$: negative-definite

when $(\star)$ holds only for $\geqslant$ or $\leqslant$: positive-semidefinite or negative-semidefinite matrix, respectively when



$x^2 + y^2$
(definite)

x^2
(semidefinite)

$x^2 - y^2$
(indefinite)

$Q > 0$:

- it is invertible
- $Q^{-1} > 0$
- all eigenvalues of Q are strictly positive
- $\exists$ a $n \times n$ real nonsingular H s. t.
$$Q = H^\top H$$
- $0 < \lambda_{min}[Q]||x||^2 \leqslant x^\top Q x \leqslant \lambda_{max}[Q]\|x\|^2, \quad \forall x \neq 0$

BIBO stability of LTI/LTV systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

Stability addresses what happens to our solutions

- do they remain bounded
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Response is due to : $\underbrace{\text{response due to } x_0}_{\text{internal stability}} + \underbrace{\text{response due to } u}_{\text{Input-output stability}}$

Def(Bounded-input-bounded-output (BIBO) stability): A system is said to be BIBO stable if every bounded input excites a bounded output (zero-state response).

An input $u(t)$ is **said to be bounded** if $u(t)$ does not grow to positive or negative infinity, or equivalently, $\exists$ a constant u_m s.t.

$$|u(t)| \leqslant u_m < \infty, \quad \forall t \geqslant 0.$$

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

$$y_f(t) = y_{zs}(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

Theorem

A SISO system $(\star)$ is BIBO if and only if $g(t)$ is absolutely integrable in $[0, \infty)$ or

$$\int_0^\infty |g(t)|dt \leqslant M < \infty$$

BIBO stability of SISO LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

$$y_f(t) = y_{zs}(t) = \int_0^t Ce^{A\,(t-\tau)}Bu(\tau)d\tau + Du(t), \qquad \bar{g}(t) := Ce^{A\,t}B$$

$$y_f(t) = y_{zs}(t) = \int_0^t \bar{g}(t-\tau)u(\tau)d\tau + Du(t) = \int_0^t \bar{g}(\tau)u(t-\tau)d\tau + Du(t)$$

Corollary

A SISO system ($\star$) is BIBO if and only if $\bar{g}(t) = Ce^{A\,t}B$ is absolutely integrable in $[0, \infty)$ or

$$\int_0^\infty |\bar{g}(t)|dt \leqslant M < \infty$$