Linear Systems I
Lecture 8

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Reading assignment: Ch 5.3, Example 5.5, Ch 3.9 and Ch. 3.11 of Ref [1]; Ch 4.2 and Ch 3.5 of Ref [1].

Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

Rest of today's lecture

- Stability of LTV and LTI systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

Stability addresses what happens to our solutions

- do they remain bounded
- will they get progressively smaller
- they diverge to infinity

Response is due to : $\underbrace{\text{response due to } x_0}_{\text{internal stability}} + \underbrace{\text{response due to } u}_{\text{Input-output stability}}$

Lets start with Internal stability:

Recall homogeneous system,

$$\dot{x} = A(t)x, x(t_0) = x_0 \in \mathbb{R}^n$$

Our solution is

$$x(t) = \phi(t, t_0)x_0, \quad t \geqslant t_0$$

Lyapunov stability. The system (LTV) is said to be

1. (*marginally*) *stable* if, for $\forall\ x_0 \in \mathbb{R}^n$, if $x(t) = \phi(t, t_0)x_0$ is uniformly bounded
2. *asymptotically stable* if, in addition, for $\forall\ x_0 \in \mathbb{R}^n$, we have $x(t) \to 0$ as $t \to \infty$,
3. *exponentially stable* if, in addition, $\exists\ c,\ \lambda > 0$, s.t. for $\forall\ x_0 \in \mathbb{R}^n$, we have

$$\|x(t)\| \leqslant ce^{-\lambda(t-t_0)}\|x_0\|, \quad \forall t \geqslant 0$$

4. *unstable* if it is not marginally stable in the Lyapunov sense.

Eigenvalue stability conditions for LTI systems

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n \Rightarrow x(t) = e^{At}x_0$$

$$J = QAQ^{-1} \quad \Longleftrightarrow \quad A = Q^{-1}JQ,$$

$$e^{At} = Q^{-1} \begin{bmatrix} e^{J_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{J_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{J_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{J_l t} \end{bmatrix} Q$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}, \qquad e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Eigenvalue conditions for Lyapunov stability of LTI systems

Consider

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

Theorem (Eigenvalue conditions) The LTI system above is

1. *marginally stable* if and only if all the eigenvalues of A have negative real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1
2. *asymptotically stable* if and only if all the eigenvalues of A have strictly negative real parts
3. *exponentially stable* if and only if all the eigenvalues of A have strictly negative real parts
4. *unstable* if and only if at least one of eigenvalues of A has a positive real part or zero real parts but the corresponding Jordan block is larger than 1×1

Internal stability of LTI systems: examples

$$A_1 = \begin{bmatrix} -1 & 3 & 4 & 5 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$\lambda = -1, -2, -2, -0.1$

Asymptotically stable

$$A_2 = \begin{bmatrix} -1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$\lambda = -1, 0, -2, -0.1$

(Marginally) stable

$$A_3 = \begin{bmatrix} -1 & 3 & 4 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$\lambda = -1, 0, 0, -0.1$

$\text{nullity}(0I - A_3) = 2$

(Marginally) stable

$$A_4 = \begin{bmatrix} -1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$\lambda = -1, 0, 0, -0.1$

$\text{nullity}(0I - A_4) = 1$

Unstable

$$A_5 = \begin{bmatrix} -1 & 3 & 4 & 5 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

$\lambda = -1, -2, -2, 0.1$

Unstable

Stability of LTV systems: examples

$$\dot{x} = A(t)x(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

Does the eigenvalue conditions for Lyapunov stability of LTI systems extend to LTV systems?

1. $$A_1(t) = \begin{bmatrix} -1 & e^{4t} \\ 0 & -5 \end{bmatrix}, \begin{cases} x_1(t) = e^{-(t-t_0)}x_1(t_0) + (t-t_0)e^{-t}(e^{5t_0}x_2(t_0)), \\ x_2(t) = e^{-5(t-t_0)}x_2(t_0) \end{cases} \quad t \geqslant t_0.$$

   Eigenvalues: -1 and -5

   Asymptotically/exponentially stable

2. $$A_2(t) = \begin{bmatrix} -2 & e^{3t} \\ 0 & -1 \end{bmatrix}, \begin{cases} x_1(t) = e^{-2(t-t_0)}x_1(t_0) + \frac{1}{4}e^{2t}e^{t_0}x_2(t_0) - \frac{1}{4}e^{-2t}e^{5t_0}x_2(t_0), \\ x_2(t) = e^{-(t-t_0)}x_2(t_0), \end{cases} \quad t \geqslant t_0$$

   Eigenvalues : -2 and -1

   Unstable

3. $$A_3(t) = \begin{bmatrix} -1 & 4e^{0.5t^2+3t} \\ 0 & -t \end{bmatrix}$$

   Eigenvalues: -1 and $-t$

   unstable