Linear Systems I
Lecture 7

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Relevant reading material: Lecture 7 of Ref[2]. Ch 3.1 to 3.4 from Ref[1]. Pages 63-70 of Ref [1] discuss the matrix exponential and its properties.

Today's lecture

- Use of Jordan/diagonalized form to computer e^{At}

Note: this note only contains parts of the in-class discussions. For more details and complete lecture overview, refer to your in-class notes.

Matrix exponential of two algebraically equivalent matrix.

- Let T be nonsingular
- Let $A = T\bar{A}T^{-1}$,

$$\boxed{e^{A\,t} = Te^{\bar{A}\,t}T^{-1}}$$

<u>Proof</u>

$$A^k = \underbrace{AAA\cdots A}_{k \text{ times}} = \underbrace{(T\bar{A}T^{-1})(T\bar{A}T^{-1})\cdots(T\bar{A}T^{-1})}_{k \text{ times}} = T\bar{A}^kT^{-1}$$

$$e^{At} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = \sum_{k=0}^{\infty}\frac{t^k}{k!}T\bar{A}^kT^{-1} = T(\sum_{k=0}^{\infty}\frac{t^k}{k!}\bar{A}^k)T^{-1} = Te^{\bar{A}\,t}T^{-1}$$

(Review of eigenvalues and eigenvectors of a matrix)

Consider a matrix $A \in \mathbb{R}^{n\times n}$,

$$Ap = \lambda p,$$

- $\lambda \in \mathbb{C}$ is eigenvalue iff we have $p \in \mathbb{C}^{n\times 1}$, $p \neq 0_{n\times 1}$
- Compute λ: $\Delta(A) = \det(\lambda I - A) = 0$; has n roots $\Rightarrow$ n eigenvalues
- Computing eigenvectors: $q \neq 0$ such that $(\lambda I - A)p = 0$, i.e., q is in the nullspace of $(\lambda I - A)$,
- Some of the properties of the eigenvectors
  - When all the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a $n \times n$ matrix A are distinct (multiplicity of all eigenvalues is 1), the nullity of $(\lambda_i I - A)$ is equal to 1. Moreover, the corresponding eigenvector set $\{p_1, \cdots, p_n\}$ is linearly independent.
  - When $\bar{\lambda}$ is an eigenvalue of A with multiplicity of $m \in [2, n]$, then we have $1 \leqslant \text{nullity}(\bar{\lambda} I - A) \leqslant m$.

Diagonalizable matrix

- An eigenvalue with multiplicity of 2 or higher is called a repeated eigenvalue.
- In contrast, an eigenvalue with multiplicity of 1 is called a simple eigenvalue.

- If A has only simple eigenvalues, it always has a diagonal form representation.
- If A has a repeated eigenvalues, then it may not have a diagonal form representation. However, it has a block-diagonal and triangular form representation.

Jordan normal form

Theorem(Jordan normal form): For every matrix $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular change of basis $\mathbf{Q} \in \mathbb{C}^{n\times n}$ that transforms A into

$$J = QAQ^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_l \end{bmatrix} = \mathrm{Diag}(J_1, J_2, J_3, \cdots, J_l),$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i\times n_i}$$

Attention: There can be several Jordan blocks for the same eigenvalue, but in that case there must be more than one independent eigenvector for that eigenvalue.

- λ_i is an eigenvalue of A
- l, number of Jordan blocks: total number of linearly independent eigenvectors of A
- J is unique up to a reordering of the Jordan blocks
- J is called Jordan normal form of A

Jordan normal form

Theorem(Jordan normal form): For every matrix $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular change of basis $\mathbf{Q} \in \mathbb{C}^{n\times n}$ that transforms A into

$$J = QAQ^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_l \end{bmatrix} = \mathrm{Diag}(J_1, J_2, J_3, \cdots, J_l),$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i\times n_i}$$

- **For every eigenvalue λ_i of A, there is at least one Jordan block associated with**
- **The number of Jordan block associated with each λ_i of A is equal to the nullity of $(A - \lambda_i I)$.**
- **If λ_j is an eigenvalue with multiplicity of $m_j = 1$, the Jordan block associated with it is $J_j = \lambda_j$**

Matrix eigenvalues and eigenvectors: diagonalizable matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(A) = \lambda(\lambda+1)(\lambda-2) = 0$$

— — — — — — — — — — — — —

$\lambda_1 = -1: \ (A-(-1)I)p_1 = 0,$
$\lambda_2 = 0: \ (A-0I)p_2 = 0,$
$\lambda_3 = 2: \ (A-2I)p_3 = 0,$
linearly independent $\{p_1, p_2, p_3\}$

$$P = \left[\begin{array}{c|c|c} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$$A = P \underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

— — — — — — — — — — — — —

λ's are district

- the matrix is diagonalizable
- the Jordan form is a diagonal matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(A) = (\lambda+1)(\lambda-2)^2 = 0$$

— — — — — — — — — — — — —

$\lambda_1 = -1: \ (A-(-1)I)p_1 = 0,$

$$\begin{cases} \lambda_2 = 2, \text{ with } m_2 = 2, \\ \text{note that nullity}(A-2I) = 2, \text{ therefore} \\ \text{ two linearly independent eigenvectors exist for } \lambda_2 : \\ (A-2I)p_2 = 0, \quad (A-2I)p_3 = 0, \end{cases}$$

linearly independent $\{p_1, p_2, p_3\}$

$$P = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ and } A = P \underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

— — — — — — — — — — — — —

Recall that

- The number of Jordan block associated with each λ_i of A is equal to the nullity of $(A-\lambda_i I)$.

if for every λ_i with multiplicity $m_i \geqslant 1$, we have nullity$(A-\lambda I) = m_i$

- the matrix is diagonalizable
- the Jordan form is a diagonal matrix

- An eigenvalue with multiplicity of 2 or higher is called a repeated eigenvalue.
- In contrast, an eigenvalue with multiplicity of 1 is called a simple eigenvalue.

- If A has only simple eigenvalues, it always has a diagonal form representation.
- If A has a repeated eigenvalues, then it may not have a diagonal form representation. However, it has a block-diagonal and triangular form representation.

Def. (Semisimple) A matrix is called semi-simple or diagonalizable if its Jordan normal form is diagonal.

— — — — — — — — — — — — — — —

Theorem Fo an $n \times n$ matrix A, the following statements are equivalent:

- A is semi-simple.
- A has n linearly independent eigenvectors.
- For any λ_i of A with multiplicity of m_i, we have nullity$(\lambda_i I - A) = m_i$.

— — — — — — — — — — — — — — —

Matrix eigenvalues and eigenvectors: Jordan normal form

A a 5×5 matrix with a simple eigenvalue λ_1, and λ_2 with multiplicity of $m = 4$

$$\exists \text{ invertible } Q : \qquad J = Q^{-1}AQ$$

$$J = \left[\begin{array}{c|c|c|c|c} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 4$

$$J = \left[\begin{array}{c|c|ccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 2$

$$J = \left[\begin{array}{c|c|c|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 3$

$$J = \left[\begin{array}{c|cccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 1$

$$J = \left[\begin{array}{c|cc|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

$\text{nullity}(\lambda_2 I - A) = 2$

One of the methods to determining the Jordan normal form

1. Compute eigenvalues of A
2. List all possible Jordan normal forms that are compatible with the eigenvalues of A:
   - eigenvalues with multiplicity equal to 1 must always correspond to 1×1 Jordan blocks
   - eigenvalues with multiplicity equal to 2 can correspond to one 2×2 block or two 1×1 blocks
   - eigenvalues with multiplicity equal to 3 can correspond to one 3×3 block , one 2×2 and two 1×1 blocks, or three 1×1 blocks, etc.
3. For each candidate Jordan normal form, check wether there exists a nonsingular matrix Q for which $J = QAQ^{-1}$. To find out wether this is so, you may solve the (equivalent, but simpler) linear equation

$$JQ = QA$$

for the unknown matrix Q and check wether it has a nonsingular solutions.

Computation of e^{At} using the Jordan normal form of A

$$J = QAQ^{-1} \quad \Longleftrightarrow \quad A = Q^{-1}JQ,$$

$$A^k = \underbrace{AAA\cdots A}_{k \text{ times}} = \underbrace{Q^{-1}JQQ^{-1}JQ\cdots Q^{-1}JQ}_{k \text{ times}} = Q^{-1}J^kQ$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = Q^{-1}\sum_{k=0}^{\infty} \frac{t^k}{k!}J^kQ =$$

$$Q^{-1}\begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}J_1^k & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}J_2^k & 0 & \cdots & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}J_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{t^k}{k!}J_l^k \end{bmatrix} Q$$

$$= Q^{-1}\begin{bmatrix} e^{J_1t} & 0 & 0 & \cdots & 0 \\ 0 & e^{J_2t} & 0 & \cdots & 0 \\ 0 & 0 & e^{J_3t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{J_lt} \end{bmatrix} Q$$

Computation of e^{At} using the Jordan normal form of A

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

Claim: $$\boxed{e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}$$

How can we verify the claim made above?

Matrix exponential of Jordan blocks

Verification: we show that e^{J_i} is the transition matrix of J_i ($e^{J_i} = \phi(t,0)$) by showing that it satisfies $\begin{cases} \frac{d}{dt}\phi(t,0) = J_i\phi(t,0) \\ \phi(0,0) = I. \end{cases}$ That is

- $e^{J_i 0} = I$ (this is trivially satisfied)
- $\frac{d}{dt}e^{J_i t} = J_i e^{J_i t}$

$$\frac{d}{dt}e^{\lambda_i t} = \frac{d}{dt}e^{J_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \lambda_i e^{\lambda_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} +$$

$$e^{\lambda_i t}\begin{bmatrix} 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ 0 & 0 & 0 & \cdots & \frac{t^{n_i-4}}{(n_i-4)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda_i e^{J_i t} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} e^{J_i t} = J_i e^{\lambda_i t}.$$

Examples

$$J = \left[\begin{array}{c|c|ccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] \Rightarrow e^{Jt} = \left[\begin{array}{c|c|ccc} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ \hline 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ \hline 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

$$J = \left[\begin{array}{c|cccc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] \Rightarrow e^{Jt} = \left[\begin{array}{c|cccc} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ \hline 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} & \frac{t^3}{6}e^{\lambda_2 t} \\ 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{t^2}{2}e^{\lambda_2 t} \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

e^{At} for diagonalizable matrices: examples

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(A) = \lambda(\lambda+1)(\lambda-2) = 0$$

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 0: \ (A - 0I)p_2 = 0,$

$\lambda_3 = 2: \ (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$A \underbrace{\left[\ p_1 \mid p_2 \mid p_3\ \right]}_{P} = \left[\ p_1 \mid p_2 \mid p_3\ \right] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \left[\begin{array}{c|c|c} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right], \ P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A = P \underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

$$e^{At} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}}_{P} \left[\begin{array}{c|c|c} e^{-t} & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{2t} \end{array}\right] \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}}_{P^{-1}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{e^{2t}}{6} - \frac{2e^{-t}}{3} + \frac{1}{2} & \frac{2e^{-t}}{3} + \frac{e^{2t}}{3} & \frac{2e^{2t}}{3} - \frac{2e^{-t}}{3} \\ \frac{e^{-t}}{3} + \frac{e^{2t}}{6} - \frac{1}{2} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{-t}}{3} + \frac{2e^{2t}}{3} \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(A) = (\lambda+1)(\lambda-2)^2 = 0$$

$\lambda_1 = -1: \ (A - (-1)I)p_1 = 0,$

$\lambda_2 = 2$, **with** $m_2 = 2$,
note that nullity$(A - 2I) = 2$, **therefore**
two linearly independent eigenvectors exist for λ_2:
$(A - 2I)p_2 = 0, \quad (A - 2I)p_3 = 0,$

linearly independent $\{p_1, p_2, p_3\}$

$$A \underbrace{\left[\ p_1 \mid p_2 \mid p_3\ \right]}_{P} = \left[\ p_1 \mid p_2 \mid p_3\ \right] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = P \underbrace{\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]}_{J \text{ with } Q = P^{-1}} P^{-1}$$

$$e^{At} = \underbrace{\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{P} \left[\begin{array}{c|c|c} e^{-t} & 0 & 0 \\ \hline 0 & e^{2t} & 0 \\ \hline 0 & 0 & e^{2t} \end{array}\right] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ e^{2t} - e^{-t} & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$