Linear Systems I
Lecture 15

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(A, B) uncontrollable: $\operatorname{rank} \mathcal{C} = \operatorname{rank}[B \; AB \; A^2B \; \cdots \; A^{n-1}B] = m < n$

$$\exists T \text{ (invertible)} : \begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{\bar{B}} u$$

$$\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B$$

Definition (Stabilizable LTI system)

Def. (Stabilizable system): The pair (A,B) is stabilizable if it is algebraically equivalent to a system in the standard form for uncontrollable systems with $n = m$ (i.e, A_u does not exist) or with A_u a stability matrix.

Definition (Stabilizable LTI system (alternative definition))

The pair (A, B) is stabilizable if there exists a state feedback gain matrix K for which all the eigenvalues of $A - BK$ have strictly negative real part.

Tests to check stabilizability of LTI systems

There are various stabilizability tests. Following are some of them:

Theorem

The following statements are equivalent:

- *The pair* (A, B) *is stabilizable;*
- *There exists no left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to the columns of* B*;*

$$\begin{cases} v^\star A = \lambda v & (Re[\lambda(A)] \geqslant 0) \\ v^\star B = 0 \end{cases} \implies v = 0$$

- $\mathrm{rank}[\lambda I - A \quad B] = n$ *for all* $Re[\lambda(A)] \geqslant 0$.

Review: tests to check stabilizability of LTI systems: example

$$\dot{x} = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} x + \begin{bmatrix} 10 \\ 4 \end{bmatrix} u$$

Controllability text:

$$\text{rank}\,\mathcal{C} = \text{rank}[B \quad AB] = \text{rank} \begin{bmatrix} 10 & 10 \\ 4 & 4 \end{bmatrix} = 1 \Longrightarrow (A, B) \text{ is not controllable!}$$

PBH eigenvalue test for controllability

- first find $\lambda[A]$:
  $\Delta(A) = \det(\lambda I - A) = (\lambda + 11)(\lambda - 11) + 120 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda[A] = \{-1, 1\}$
- check rank of $[\lambda I - A \;\; B]$ for $\lambda[A] = \{-1, 1\}$

$$\lambda = -1: \;\text{rank}[-I - A \;\; B] = \text{rank} \begin{bmatrix} 10 & -30 & 10 \\ 4 & -12 & 4 \end{bmatrix} = 1 \Rightarrow \lambda = -1 \text{ is } \underline{\text{not}} \text{ a controllable eigenvalue}$$

$$\lambda = 1: \;\text{rank}[I - A \;\; B] = \text{rank} \begin{bmatrix} 12 & -30 & 10 \\ 4 & -10 & 4 \end{bmatrix} = 2 \Rightarrow \lambda = 1 \text{ is a controllable eigenvalue}$$

Then:

- (A,B) is not controllable
- (A,B) is stablilizable: because $\text{rank}[\lambda I - A \;\; B] = n$ for the eigenvalue with positive real part $\lambda = 1$.

Regulation via state-feedback control when (A,B) is controllable: pole-placement/eigenvalue placement

$$\dot{x} = Ax + Bu, \quad u = -Kx \Rightarrow \dot{x} = (A - BK)x$$

- (A, B) **controllable: Given any symmetric set of** n **complex numbers** $\{\nu_1, \nu_2, \cdots, \nu_n\}$**, there exists a full-state feedback matrix** K **such that the closed-loop system matrix** $(A - BK)$ **has eigenvalues equal to these** ν_i**'s.**

$$\exists K: \ \mathbf{det}(\lambda I - (A - BK)) = \underbrace{(\lambda - \nu_1)(\lambda - \nu_2)\cdots(\lambda - \nu_n)}_{\text{desired charac. polynomial}}$$

  - **Single input systems (**$u \in \mathbb{R}$, $B \in \mathbb{R}^{n\times 1}$**): See HW 6 for a procedure for eigenvalue placement. You can also use Achermann formula.**

  - **Multi-input systems (**$u \in \mathbb{R}^p$, $B \in \mathbb{R}^{n\times p}$**): Theorem below gives a solution**

    Theorem: Suppose ($A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}$***) is controllable. Then (***$A + BF, Bv$***), is controllable for almost any*** $F \in \mathbb{R}^{p\times n}$ ***and*** $v \in \mathbb{R}^{p\times 1}$.

    State feedback for multi input systems: Example, place the eigenvalues at $\{-1, -2, -3\}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

    - **choose** $F \in \mathbb{R}^{p\times n}$ **and** $v \in \mathbb{R}^{p\times 1}$ **such that** $(A + BF, Bv)$ **is controllable**

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow \bar{A} = A + BF = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 5 & 1 \end{bmatrix}, \bar{B} = Bv = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{rank}(\bar{\mathcal{C}}) = \text{rank} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 2 & 8 & 25 \end{bmatrix}$$

    - **place eigenvalues of** $(\bar{A}, \bar{B})$ **at your desired location** $\{-1, -2, -3\}$ **using the methods for single input systems: here I get** $\bar{K} = [28 \ 80 \ -8]$

    - **the** K **in** $A - BK$ **is obtained from** $A - BK = A + BF - Bv\bar{K} = A - B\underbrace{(-F + v\bar{K})}_{K}$**, which gives**

$$K = \begin{bmatrix} 27 & 80 & -8 \\ 0 & -1 & 0 \end{bmatrix}$$

    - **For this example** $A - BK = \begin{bmatrix} -26 & -80 & 8 \\ 1 & 3 & 0 \\ -50 & -155 & 17 \end{bmatrix}$**, with eigenvalues at the desired location** $\{-1, -2, -3\}$.

Regulation via state-feedback control when (A,B) is stabilizable: pole-placement/eigenvalue placement

$$\dot{x} = Ax + Bu, \quad u = -Kx \Rightarrow \dot{x} = (A - BK)x$$

- (A, B) is not controllable: $\text{rank}(\mathcal{C}) = m < n$ $(A \in \mathbb{R}^{n\times n})$:

$$\exists T \text{invertible} : x = T\bar{x} : \quad \dot{\bar{x}} = \begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$u = -Kx = -KT\bar{x} = -\bar{K}\bar{x} = -\begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix}$$

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_c - B_c\bar{K}_1 & A_{12} - B_c\bar{K}_2 \\ 0 & A_u \end{bmatrix}}_{\bar{A} - \bar{B}\bar{K} = T^{-1}(A - BK)T} \begin{bmatrix} x_c \\ x_u \end{bmatrix}$$

(A_c, B_c) is controllable, we can place eigenvalues of $(A_c - B_c\bar{K}_1)$ in any location we want using state feedback!

We can only change the location of controllable eigenvalues using state feedback

We can only stabilize a system whose uncontrollable eigenvalues are stable

1. Transfer (A, B) to the controllable decomposition form
2. Recall that $\text{eig}(A) = \text{eig}(A_c) \cup \text{eig}(A_u)$
3. (A_c, B_c) is controllable, so you can place the eigenvalues of this controllable part at your desired locations using gain $\bar{k}_1$, i.e., eigenvalues of $A_c - B_c\bar{k}_1$
4. you can find the gain K placing the controllable eigenvalues in your desired places using $K = \bar{K}T^{-1}$, with $\bar{K} = \begin{bmatrix} \bar{k}_1 & \bar{k}_2 \end{bmatrix}$. You can set k_2 to zero.
5. you should arrive at $\text{eig}(A - BK) = \text{eig}(A_c - B_c\bar{k}_1) \cup \text{eig}(A_u)$

State feedback design for a stabilizable system: example

$$\dot{x} = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} x + \begin{bmatrix} 10 \\ 4 \end{bmatrix} u$$

Controllability text:

$$\text{rank}\, \mathcal{C} = \text{rank}[B \quad AB] = \text{rank} \begin{bmatrix} 10 & 10 \\ 4 & 4 \end{bmatrix} = 1 \Longrightarrow (A, B) \text{ is not controllable!}$$

eig(A): $\Delta(A) = \det(\lambda I - A) = (\lambda + 11)(\lambda - 11) + 120 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \text{eig}(A) = \{-1, 1\}$

Controllable decomposition

$$T = \begin{bmatrix} 5 & 0 \\ 2 & 0.2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & -1.2 \\ 0 & -1 \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Objective Place eigenvalues of $A - BK$ at $\{-1, -3\}$

We use $(A_c, B_c) = (1, 2)$ to place eigenvalue of the controllable part at -3: $\lambda(A_c - B_c \bar{k}_1) = -3$

$$\lambda - (1 - 2\bar{k}_1) = \lambda + 3 \;\Rightarrow \bar{k}_1 = 2.$$

$$u = \bar{K}\bar{x} = [2 \quad 0]\,\bar{x} = [2 \quad 0]\,T^{-1}x = [2 \quad 0] \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix} x = \underbrace{[0.4 \quad 0]}_{K}\, x.$$

You can confirm this by checking eigenvalues of $A - BK = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix} [0.4 \quad 0] = \begin{bmatrix} -15 & 30 \\ -5.6 & 11 \end{bmatrix}.$

State feedback design for a stabilizable system: example (alternative approach)

Consider the state feedback $u = -Kx = -[k_1 \;\; k_2]x$

$$A_{cl} = A - BK = \begin{bmatrix} -11 - 10k_1 & 30 - 10k_2 \\ -4 - 4k_1 & 11 - 4k_2 \end{bmatrix}$$

$$\Delta(A_{cl}) = \Delta(A{-}BK) = \det\left(\lambda I - \begin{bmatrix} -11 - 10k_1 & 30 - 10k_2 \\ -4 - 4k_1 & 11 - 4k_2 \end{bmatrix}\right) = (\lambda+1)(\lambda+10k_1+4k_2-1) = 0$$

$$\text{eig}(A_{cl}) = \{-1, -10k_1 - 4k_2 + 1\}$$

- Notice that we cannot change the location of uncontrollable eigenvalue but we can put the controllable eigenvalue in any new location using state feedback!
- We can pick k_1 and k_2 such that A_{cl} has eigenvalues with strictly negative real parts and, as such, stabilize the closed-loop system using $u = -Kx$.
- For example $k_1 = 0$ and $k_2 = 1$ results in $\lambda[A_{cl}] = \{-1, -3\}$.
  You can confirm this by checking eigenvalues of
  $A - BK = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix} [0 \;\; 1] = \begin{bmatrix} -11 & 20 \\ -4 & 7 \end{bmatrix}$.

Next Lecture

Observability of LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n, \ u \in \mathbb{R}^p \\ y = Cx + Du, & y \in \mathbb{R}^q \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

Question of interest in Observability: Can we reconstruct $x(0)$ by knowing $y(t)$ and $u(t)$ over some finite time interval $[0, t_1]$? (By knowing the initial condition, we can reconstruct the entire state $x(t)$, then use it in our state feedback to control the system)

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Observability of LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \\ y = Cx + Du, & y \in \mathbb{R}^q \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

Question of interest in Observability: Can we reconstruct $x(0)$ by knowing $y(t)$ and $u(t)$ over some finite time interval $[0, t_1]$? (By knowing the initial condition, we can reconstruct the entire state $x(t)$, then use it in our state feedback to control the system)

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \Leftrightarrow \bar{y}(t) = Ce^{At}x(0)$$

$$\bar{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

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The LTI state-space equation $(\star)$ is said **to be observable** if for any unknown initial state $x(0)$, $\exists$ finite time $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.

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Tests for Observability of LTI systems

The following statements are equivalent:

1. the n-dimentional pair (A, C) is observable
2. The $n \times n$ matrix $\boxed{W_o(t) = \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau}$ is nonsingular for all $t > 0$.
3. Let $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nq \times n}$ be the observability matrix, then $\text{rank}(\mathcal{O}) = n$
4. rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all complex λ
5. rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all λ eigenvalues of A
6. If in addition, all eigenvalues of A have negative real parts, then the unique solution of
$$A^\top W_o + W_o A = -C^\top C$$
is positive definite. The solution is called the observability Gramian and can be expressed as
$$W_o = \int_0^\infty e^{A^\top \tau} C^\top C e^{A\tau} d\tau$$

Review of controllable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

Theorem

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} C_u & C_u \end{bmatrix}, \quad \bar{D} = D,$$
$$A_c \in \mathbb{R}^{m \times m}, \quad B_c \in \mathbb{R}^{m \times p}, \quad C_c \in \mathbb{R}^{q \times m},$$

$$T = [\; \underbrace{t_1 \quad t_2 \quad \cdots \quad t_m}_{} \;\big|\; \underbrace{t_{m+1} \quad t_{m+2} \quad \cdots \quad t_n}_{} \;]$$

$\{$ m linearly independent columns of $\mathcal{C}$

$\{$ any way you can s.t. all columns of T are linearly independent

(A_c, B_c) is controllable!

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_c(sI - A_c)^{-1}B_c + D$$

Observable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

Theorem

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n :$$

$\exists\, T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_o & 0 \\ A_{12} & A_{\bar{o}} \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}$$
$$\bar{C} = CT = \begin{bmatrix} C_o & 0 \end{bmatrix}, \quad \bar{D} = D,$$
$$A_o \in \mathbb{R}^{\bar{m} \times \bar{m}}, \quad B_o \in \mathbb{R}^{\bar{m} \times p}, \quad C_o \in \mathbb{R}^{q \times \bar{m}},$$

$$T = [\ \underbrace{t_1 \quad t_2 \quad \cdots \quad t_{\bar{m}}} \ \mid \ \underbrace{t_{\bar{m}+1} \quad t_{\bar{m}+2} \quad \cdots \quad t_n}\ ]$$

- $t_1 \ldots t_{\bar{m}}$: any way you can s.t. all columns of T are linearly independent
- $t_{\bar{m}+1} \ldots t_n$: $n - \bar{m}$ linearly independent vectors spanning the nullspace of $\mathcal{O}$

(A_o, B_o) is observable.

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_o(sI - A_o)^{-1}B_o + D$$

Kalman decomposition

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, y \in \mathbb{R}^q$$

Theorem

$\exists T$ invertible s.t. $\bar{x} = T^{-1}x$ transforms state equations to

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n$$

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & {}_{xo} & 0 \\ A_{cx} & A_{c\bar{o}} & A_{xx} & A_{x\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}x} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{\bar{C} = CT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

$$T = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$$

- columns of $[T_{co} \;\; T_{c\bar{o}}]$ span the Im$\mathcal{C}$
- columns of $T_{c\bar{o}}$ span the null$\mathcal{O} \cap$ Im$\mathcal{C}$
- columns of $[T_{c\bar{o}} \;\; T_{\bar{c}\bar{o}}]$ span the null$\mathcal{O}$
- columns of $T_{\bar{c}o}$ along with the elements described above construct an invertible T

- (A_{co}, B_{co}, C_{co}) is both controllable and observable.
- $\left(\begin{bmatrix} A_{co} & 0 \\ A_{cx} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix}\right)$ is controllable
- $\left(\begin{bmatrix} A_{co} & A_{xo} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix}\right)$ is controllable

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_{co}(sI - A_{co})^{-1}B_{co} + D$$