Linear Systems I
Lecture 13

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Nov. 10, 2015

Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

This lecture

- Controllability of LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

  - Can we steer the system states from every point in $\mathbb{R}^n$ to every other point in $\mathbb{R}^n$ in finite time? ((completely-state) controllable system)
    - test to evaluate controllability

Review: Completely-state controllable and reachable LTV systems

$$\dot{x} = A(t)x + B(t)u, \quad x \in \mathbb{R}^n$$

Definition ((Completely-state) reachable system)

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 = 0$,

$$\left\{x_1 \in \mathbb{R}^n : \exists u(.), x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau\right\} = \mathbb{R}^n$$

Definition ((Completely-state) controllable system)

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 \neq 0$,

$$\left\{x_0 \in \mathbb{R}^n : \exists u(.), 0 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau\right\} = \mathbb{R}^n$$

or

$$\left\{x_0 \in \mathbb{R}^n : \exists v(.) = -u(.), x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)v(\tau)d\tau\right\} = \mathbb{R}^n$$

Review: controllability matrix for LTI systems

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

Definition (Reachability and controllability gramians for given $t_1 > t_0 \geqslant 0$)

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^\top \phi(t_1, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_1-\tau)} BB^\top e^{A^\top(t_1-\tau)} d\tau = \int_0^{t_1-t_0} e^{A\tau} BB^\top e^{A^\top \tau} d\tau,$$

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^\top \phi(t_0, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^\top e^{A^\top(t_0-\tau)} d\tau = \int_0^{t_1-t_0} e^{-A\tau} BB^\top e^{-A^\top \tau} d\tau.$$

Theorem

Let

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times (np)}.$$

For any two time $t_1 > t_0 \geqslant 0$

$$\mathcal{R}[t_0, t_1] = Im W_R(t_0, t_1) = Im \mathcal{C} = Im W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Controllable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

Theorem

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $x = T\bar{x}$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$$A_c \in \mathbb{R}^{m\times m}, \quad , B_c \in \mathbb{R}^{m\times p}, \quad A_u \in \mathbb{R}^{(n-m)\times(n-m)}, \quad A_{12} \in \mathbb{R}^{m\times(n-m)}.$$

Controllable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

Theorem

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$$A_c \in \mathbb{R}^{m \times m}, \quad , B_c \in \mathbb{R}^{m \times p}, \quad A_u \in \mathbb{R}^{(n-m)\times(n-m)}, \quad A_{12} \in \mathbb{R}^{m \times (n-m)}.$$

Corollary

- *The pair* (A_c, B_c) *is controllale, i.e.,* $\text{rank}\begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{m-1}B_c \end{bmatrix} = m$
- *The controllable subspace of* $(\bar{A}, \bar{B})$ *is Im* $\begin{bmatrix} I_{m \times m} \\ 0_{(n-m)\times m} \end{bmatrix}$

Controllable decomposition: example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$\mathcal{C}$ has only two linearly independent columns: $A^2B = -2\,B - 3\,AB$

Controllable decomposition

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 7 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array}\right], \quad \bar{B} = T^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right]$$

$$A_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Controllable decomposition: transfer function

Theorem

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + D$$

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $x = T\bar{x}$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0_{(n-m)\times m} & A_u \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_c \\ 0_{(n-m)\times p} \end{bmatrix}$$
$$\bar{C} = CT = \begin{bmatrix} C_c & C_u \end{bmatrix}, \quad \bar{D} = D$$

For (A,B,C,D): $\hat{G}(s) = C(sI - A)^{-1}B + D$.

Transfer function of two algebraically equivalent system is the same

$$\hat{G}(s) = \hat{\bar{G}}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + D = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A_c) & -A_{12} \\ 0 & (sI - A_u) \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D$$

$$= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} & \times \\ 0 & (sI - A_u)^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D = C_c(sI - A_c)^{-1}B_c + D.$$

$$\hat{G}(s) = C_c(sI - A_c)^{-1}B_c + D$$

Transfer function of an LTI system is equal to the transfer function of its controllable part.

Controllable decomposition: example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$\mathcal{C}$ has only two linearly independent columns: $A^2B = -2\,B - 3\,AB$

Controllable decomposition

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 7 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array}\right], \quad \bar{B} = T^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right], \quad \bar{C} = CT = \left[\begin{array}{cc|c} 1 & -2 & 0 \end{array}\right]$$

$$A_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{G}(s) = C_c(sI - A_c)^{-1}B_c + D = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s-2}{(s+1)(s+2)}$$

Popov-Belevitch-Hautus (PBH) test for controllability

Theorem (Eigenvector test)

(A, B) *is controllable iff there exists no left eigenvector of A orthogonal to the columns of* B*., i.e.,*

$$\begin{cases} v^\star A = \lambda v^\star, \\ v^\star B = 0, \end{cases} \implies v = 0$$

or

$$\begin{cases} A^\top v = \lambda v, \\ B^\top v = 0, \end{cases} \implies v = 0$$

Theorem (Eigenvalue test)

(A, B) *is controllable iff* $\operatorname{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}$*.*
or
(A, B) *is controllable iff* $\operatorname{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$ *for all* λ *eigenvalue of* A*.*

A part of proof of eigenvector PBH test for controllability

$$\left(\text{if } \begin{cases} v^\star A = \lambda v^\star, \\ v^\star B = 0, \end{cases} \quad \text{then } \ v = 0 \right) \Longrightarrow (A, B) \text{ controllable}$$

By contradiction: Let $(v^\star A = \lambda v^\star,\ v^\star B = 0)$ be only true for $v = 0_{n\times 1}$. Assume (A, B) is not controllable, i.e., rank $\mathcal{C} < n$.

$$\exists T \text{ invertible: } : \bar{A} = T^{-1} A T = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = T^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

Take any λ eigenvalue of A_u and its associated left eigenvector v_2, i.e.,

$$v_2 \neq 0,\ v_2^\star A_u = \lambda v_2^\star$$

Define $v := T^{-\top} \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \neq 0$ (Note: $v^\star = \begin{bmatrix} 0 & v_2^\star \end{bmatrix} T^{-1} \neq 0$).

Next, we show that v is a left eigenvector of A (recall that $A = T\bar{A}T^{-1}$):

$$v^\star A = v^\star (T\bar{A}T^{-1}) = \begin{bmatrix} 0 & v_2^\star \end{bmatrix} T^{-1} \left(T \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} T^{-1} \right) =$$

$$\begin{bmatrix} 0 & v_2^\star A_u \end{bmatrix} T^{-1} = \begin{bmatrix} 0 & \lambda v_2^\star \end{bmatrix} T^{-1} = \lambda v^\star$$

But

$$v^\star B = \begin{bmatrix} 0 & v_2^\star \end{bmatrix} T^{-1} \left(T \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & v_2^\star \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} = 0$$

this means that $\exists v \neq 0$ such that $(v^\star A = \lambda v^\star,\ v^\star B = 0)$, which is a contradiction!

PBH test for controllability: example (Uncontrollable eigenvalues)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$\mathcal{C}$ **has only two linearly independent columns**: $A^2 B = -2B - 3AB \Rightarrow$

The system is not controllable

PBH eigenvector and eigen value controllability tests:

$$\lambda = \{-1, -2, -3\}$$

Corresponding left eigenvectors:

$$v_1 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$v_1^\top B \neq 0, \quad v_2^\top B \neq 0$$

$$v_3^\top B = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = 0$$

The system is not controllable

$$\text{rank} \begin{bmatrix} -1I - A & B \end{bmatrix} = \text{rank} \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 6 & 11 & 5 & -3 \end{bmatrix} = 3$$

$$\text{rank} \begin{bmatrix} -2I - A & B \end{bmatrix} = \text{rank} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & -2 & -1 & 1 \\ 6 & 11 & 4 & -3 \end{bmatrix} = 3$$

$$\text{rank} \begin{bmatrix} -3I - A & B \end{bmatrix} = \text{rank} \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & -3 & -1 & 1 \\ 6 & 11 & 3 & -3 \end{bmatrix} = 2$$

The system is not controllable

$\lambda_3 = -3$ **is the uncontrollable eigenvalue**

$$\bar{A} = T^{-1} A T = \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array}\right], \quad \bar{B} = T^{-1} B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right]$$

The following material will be covered on next lecture.

Lyapunov test for controllability

Theorem (Lyapunov test for controllability)

Assume that all the eigenvalues of A have <u>*negative real parts*</u>*.* (A, B) *is controllable iff there exists a unique* $W > 0$ *which is solves*

$$AW + WA^\top = -BB^\top$$

Moreover this solution is

$$W = \int_0^\infty e^{A^\top \tau} BB^\top e^{A^\top \tau} d\tau$$

Proof: ....

Consider

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

Theorem: The following five conditions are equivalent for the **LTI** system above

1. The system is asymptotically stable
2. The system is exponentially stable
3. All the eigenvalues of A have strictly negative real parts
4. For every $Q > 0$, $\exists$ a unique solution P for the following Lyapunov equation

$$A^\top P + PA = -Q$$

   Moreover P is symmetric and positive definite.

5. $\exists$ $P > 0$ for which the following Lyapunov matrix inequality holds

$$A^\top P + PA < 0$$

6. For every matrix $\bar{B}$ for which $(A, \bar{B})$ is controllable, there exists a unique solution $P > 0$ to the Lyapunov

$$AP + PA^\top = \bar{B}\bar{B}^\top$$

   Moreover, P is symmetric and positive definite, and $P = \int_0^\infty e^{A^\top \tau}\bar{B}\bar{B}^\top e^{A^\top \tau} d\tau$.

Regulation via state-feedback control

Consider

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \neq 0 \in \mathbb{R}^n$$

Definition (Regulation problem)

Starting from nonzero initial conditions, force the state vector to zero as $t \to \infty$.

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Goal: We want to solve this problem using state feedback $u = -Kx$

$$\dot{x} = A_{cl}x, \quad A_{cl} = (A - BK) \in \mathbb{R}^{n\times n}, K \in \mathbb{R}^{n\times p},$$
$$x(0) = x_0 \neq 0 \in \mathbb{R}^n.$$

– –

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- A is Hurwitz, regulation can be solved using $u = 0$
- We want some performance
  - how fast
  - certain transient response
  - minimum energy,
  - etc

Regulation via state-feedback control

— —

Goal: We want to solve this problem using state feedback $u = -Kx$

$$\dot{x} = A_{cl}x, \quad A_{cl} = (A - BK) \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{n \times p},$$
$$x(0) = x_0 \neq 0 \in \mathbb{R}^n.$$

— —

Regulation via full state feedback:

- fast with rate $\mu > 0$: place the eigenvalues such that $-\mathrm{Re}(\lambda) \leqslant \mu$
- control over transient: place eigenvalues in certain locations

Regulation via state-feedback control when (A,B) is controllable

Theorem

Let (A, B) *be controllable. For every* $\mu > 0$, *it is possible to find a state-feedback controller* $u = -Ku$ *that places all the eigenvalues of the closed-loop matrix* $(A - BK)$ *on the complex semi plain Re*$[s] \leqslant -\mu$.

Theorem (Eigenvalue assignment)

Let (A, B) *be controllable. Given any set of* n *complex numbers* $\lambda_1, \lambda_2, \cdots, \lambda_n$, *there exists a full-state feedback matrix* K *such that the closed-loop system matrix* $(A - BK)$ *has eigenvalues equal to these* λ_i*'s.*