Linear Systems I
Lecture 11

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Complementary Reading: Ch 6.1, 6.2 and 6.8 from Ref[1].

Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

- Controllable and reachable subspaces for LTV systems of the form

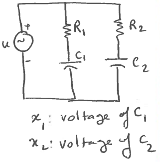
$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

  - Can we steer the system states from zero initial conditions to any place in the space in finite time? If not for all the points, what subset of space we can reach in finite time?
  - Can we steer the system states from any arbitrary point in the space to the origin in finite time? If not for all the points, what subset of the space we can steer to origin in finite?

- Special case: controllable and reachable subspaces for LTI systems of the form

$$\begin{cases} \dot{x} = Ax + B\,u, \\ y = Cx + D\,u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{bmatrix} u$$



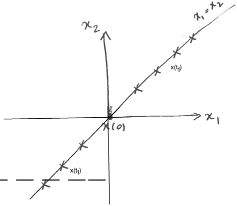
$$x(t) = \begin{bmatrix} e^{-\frac{t}{R_1 C_1}} x_1(0) \\ e^{-\frac{t}{R_2 C_2}} x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{e^{-\frac{t-\tau}{R_1 C_1}}}{R_1 C_1} \\ \frac{e^{-\frac{t-\tau}{R_2 C_2}}}{R_2 C_2} \end{bmatrix} u(\tau) d\tau$$

when $R_1 C_1 = R_2 C_2 = 1/\omega$: $x(t) = e^{-\omega t} x(0) + \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x(0) = 0$

$$x(t_1) = \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha(t_1) := \omega \int_0^{t_1} e^{-\omega(t-\tau)} u(\tau) d\tau$$

$$\mathcal{X}_1 = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$$

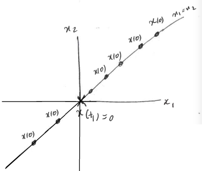


$x(0) = x_0 \to x(t_1) = 0$

$$0 = e^{-\omega t} x(0) + \omega \int_0^{t_1} e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

possible if $x(0)$ is aligned with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$\mathcal{X}_0 = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$$



Controllable and reachable subspaces for LTV systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau \Rightarrow$$

$$\text{at } t = t_1 : \quad x_1 = x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Definition (Reachable subspace (controllable-from-the-origin))

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 = 0$,

$$\mathcal{R}[t_0, t_1] := \left\{ x_1 \in \mathbb{R}^n : \exists u(.), x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau \right\}$$

Definition (Controllable subspace (controllable-to-the-origin))

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 \neq 0$,

$$\mathscr{C}[t_0, t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists u(.), 0 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau \right\}$$

$$\mathscr{C}[t_0, t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists v(.) = -u(.), x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)v(\tau)d\tau \right\}$$

Controllable and reachable subspaces: example

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

$$x(t) = \begin{bmatrix} e^{-\frac{t}{R_1C_1}} x_1(0) \\ e^{-\frac{t}{R_2C_2}} x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{e^{-\frac{t-\tau}{R_1C_1}}}{R_1C_1} \\ \frac{e^{-\frac{t-\tau}{R_2C_2}}}{R_2C_2} \end{bmatrix} u(\tau)d\tau$$

when $R_1C_1 = R_2C_2 = 1/\omega$

$$x(t) = e^{-\omega t}x(0) + \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$x(0) = 0$

$$x(t) = \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha(t) := \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau$$

$$\mathcal{R}[t_0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$$

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$x(0) = x_0 \to x(t) = 0$

$$0 = e^{-\omega t}x(0) + \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

possible if $x(0)$ is aligned with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$\mathscr{C}[t_0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$$

Reachability gramians

Definition (Reachability gramian for given $t_1 > t_0 \geqslant 0$)

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^\top \phi(t_1, \tau)^\top d\tau,$$

Theorem (Reachable subspace)

Given two times $t_1 > t_0 \geqslant 0$,

$$\mathcal{R}[t_0, t_1] = \mathit{Im} W_R(t_0, t_1),$$

Moreover, if $x_1 = W_R(t_0, t_1)\eta_1 \in \mathit{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^\top \phi(t_1, t)^\top \eta_1, \quad t \in [t_0, t_1], \quad \textit{minimum-energy open-loop controller}$$

can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$.

Example

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & t \end{bmatrix} x + \begin{bmatrix} \sqrt{t} \\ \sqrt{t} \end{bmatrix} u, \quad t_0 \geqslant 0 \Rightarrow \phi(t, t_0) = \begin{bmatrix} 1 & -1 + e^{\frac{t^2 - t_0^2}{2}} \\ 0 & e^{\frac{t^2 - t_0^2}{2}} \end{bmatrix}$$

Is this system reachable?

$$\phi(t_1, \tau) B(\tau) = \begin{bmatrix} 1 & -1 + e^{\frac{t_1^2 - \tau^2}{2}} \\ 0 & e^{\frac{t_1^2 - \tau^2}{2}} \end{bmatrix} \begin{bmatrix} \sqrt{\tau} \\ \sqrt{\tau} \end{bmatrix} = \begin{bmatrix} \sqrt{\tau}\, e^{\frac{t_1^2 - \tau^2}{2}} \\ \sqrt{\tau}\, e^{\frac{t_1^2 - \tau^2}{2}} \end{bmatrix}$$

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^\top \phi(t_1, \tau)^\top d\tau = \int_{t_0}^{t_1} \begin{bmatrix} \tau e^{t_1^2 - \tau^2} & \tau e^{t_1^2 - \tau^2} \\ \tau e^{t_1^2 - \tau^2} & \tau e^{t_1^2 - \tau^2} \end{bmatrix} d\tau =$$

$$-\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} e^{t_1^2 - t_0^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}(-1 + e^{t_1^2 - t_0^2}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This system is not reachable because $\det(W_R(t_0, t_1)) = 0$ for all $t_1 > t_0 \geqslant 0$.

The reachable set is $\mathcal{R}[t_0, t_1] = \mathrm{Im} W_R(t_0, t_1) = \mathrm{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Find the controller to take the system from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $(t_1 = 1)$.

$$x_1 = W_R(t_0, t_1)\eta_1 \in \mathrm{Im} W_R(t_0, t_1) \Rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} = (\frac{1}{2}(-1 + e^1) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \eta_1 \Rightarrow \eta_1 = \frac{4}{e-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u(t) = B(t)^\top \phi(1, t)^\top \eta_1 = \frac{4}{e-1} \begin{bmatrix} \sqrt{t}\, e^{\frac{1-t^2}{2}} & \sqrt{t}\, e^{\frac{1-t^2}{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{4}{e-1} \sqrt{t}\, e^{\frac{1-t^2}{2}}, \quad t \in [0, 1].$$

Controllability gramians

Definition (Controllability gramians for given $t_1 > t_0 \geqslant 0$)

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^\top \phi(t_0, \tau)^\top d\tau,$$

Theorem (Controllable subspace)

Given two times $t_1 > t_0 \geqslant 0$,

$$\mathscr{C}[t_0, t_1] = \mathit{Im} W_C(t_0, t_1),$$

Moreover, if $x_0 = W_C(t_0, t_1)\eta_0 \in \mathit{Im} W_C(t_0, t_1)$, *the control*

$$u(t) = -B(t)^\top \phi(t_0, t)^\top \eta_0, \quad t \in [t_0, t_1],$$ *minimum-energy open-loop controller*

can be used to transfer the state from $x(t_0) = x_0$ *to* $x(t_1) = 0$.

Controllability matrix for LTI systems

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

Definition (Reachability and controllability gramians for given $t_1 > t_0 \geqslant 0$)

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^\top \phi(t_1, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_1 - \tau)} B B^\top e^{A^\top (t_1 - \tau)} d\tau,$$

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^\top \phi(t_0, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_0 - \tau)} B B^\top e^{A^\top (t_0 - \tau)} d\tau,$$

Theorem

Let

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times (np)}.$$

For any two time $t_1 > t_0 \geqslant 0$

$$\mathcal{R}[t_0, t_1] = \mathit{Im} W_R(t_0, t_1) = \mathit{Im}\mathcal{C} = \mathit{Im} W_C(t_0, t_1) = \mathscr{C}[t_0, t_1].$$

- The controllable and reachable subspaces are the same for continuous-time LTI systems. Because of this for continuous-time LTI systems one simply studies controllability and neglects reachability.

Controllable and reachable subspaces: example

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

This is an LTI system, therefore the controllable and reachable subsets are equal to one and other and can be obtained from finding Image (range) of controllability matrix:

- $\omega = \frac{1}{R_1C_1} = \frac{1}{R_2C_2}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \omega & -\omega^2 \\ \omega & -\omega^2 \end{bmatrix}$$

$\mathcal{C}$ has one linearly independent column. The reachable and controllable subsets are ($\alpha \in \mathbb{R}$):

$$\mathrm{Im}\mathcal{C} = \alpha \begin{bmatrix} \omega \\ \omega \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathcal{R}(t_0, t_1) = \mathscr{C}(t_0, t_1)$$

- $\omega_1 = \frac{1}{R_1C_1} \neq \omega_2 = \frac{1}{R_2C_2}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \omega_1 & -\omega_1^2 \\ \omega_2 & -\omega_2^2 \end{bmatrix}$$

$\mathcal{C}$ has two linearly independent columns. The reachable and controllable subsets are ($\alpha, \beta \in \mathbb{R}$):

$$\mathrm{Im}\mathcal{C} = \alpha \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \beta \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \end{bmatrix} = \mathbb{R}^2 = \mathcal{R}(t_0, t_1) = \mathscr{C}(t_0, t_1)$$

In this case every point in the $\mathbb{R}^2$ is reachable from the origin in finite time and every point in the $\mathbb{R}^2$ can be steered to origin in finite time.
