MAE270A: Concepts of Controllable/Reachability for LTV Systems
Stabilizability and full state feedback design for LTI systems
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Problem setting.

Consider the linear system:

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad (1)$$

Research questions.

Is it possible to steer the states of the LTV system (1) from the starting point of zero to any point in the $\mathbb{R}^n$ space in a finite amount of time with the help of a control input $u(t)$? If not, what points in the $\mathbb{R}^n$ space can be reached in a finite amount of time from the initial condition of zero with unrestricted control inputs $u(t)$? We address these questions using the concept of **Reachability**.

Is it possible to steer the states of the LTV system (1) from any starting point $x(t_0) \in \mathbb{R}^n$ to zero in a finite amount of time with the help of a control input $u(t)$? If not, what points in the space can be directed to zero in a finite time period with unrestricted control inputs $u(t)$?We address these questions using the concept of **Controllability**.

Motivating example.

Consider the LTI system

$$\dot{x} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} x + \begin{bmatrix} b \\ b \end{bmatrix} u, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2, \qquad (2)$$

which is consisted of two parallel system $x_i = ax_i + bu$, $i \in 1, 2$. Starting from $t_0 = 0$, the trajectories of this system are given by $x(t) = \mathrm{e}^{a\,t} \begin{bmatrix} x_1(0) + b\int_0^t \mathrm{e}^{-a\,\tau} u(\tau)\mathrm{d}\tau \\ x_2(0) + b\int_0^t \mathrm{e}^{-a\,\tau} u(\tau)\mathrm{d}\tau \end{bmatrix}$. The trajectories of the system show that, at any given finite time $t_1 \in >\nvdash$, no matter what control we use,

- starting from initial condition zero, the system can only reach points that satisfy $x_1(t_1) = x_2(t_1)$, i.e., reachable set is $\mathrm{R[0,t_1]} = \left\{ x \in \mathbb{R}^2 | x = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$.
- only initial conditions $x(0) \in \mathbb{R}^2$ that satisfy $x_1(0) = x_2(0)$ can be steered to origin, i.e., controllable set is $\mathrm{C[0,t_1]} = \left\{ x \in \mathbb{R}^2 | x = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$.

Take away

Even if you have unrestricted control, some systems structurally are bound to be only partially controllable or reachable.

Definitions

- Reachable system
- Controllable system

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System (1) is said to be

- (fully) reachable if for any initial condition $x_0 = x(t_0) = 0$ and any finite state $x_1 = x(t_1) \in \mathbb{R}^n$, there exists an input $u(t)$ that transfers the states to x_1 in the finite time $t_1 - t_0$.
- (fully) controllable if for any finite initial condition $x_0 = x(t_0) \in \mathbb{R}^n$ there exists an input $u(t)$ that transfers the states to $x_1 = x(t_1) = 0$ in the finite time $t_1 - t_0$.

Research question.

Our motivating example showed us that not all systems can be steered to zero from any starting point or steered from zero to any point in the space in a finite amount of time using control inputs. The questions we ask are

- What systems are fully reachable? Put in another way, how do we determine a system is fully reachable?
- What systems are fully controllable? Put in another way, how do we determine a system is fully controllable?
- If a system is not fully reachable, what subset of the state space is reachable for this system?
- If a system is not fully controllable, what subset of the state space is reachable for this system?

Definitions

- Reachable set
- Controlable set

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To address our aforementioned questions, we need to formalize the definition of the reachable set and controllable set first. Recall that the trajectories of system (1), are given by $x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)b(\tau)u(\tau)\mathrm{d}\tau$. Trajectories of a system are the points in space that the states of the system go through. Knowing the structural form of the trajectories of the system, given a finite initial time t_0 and finite final time $t_1 > t_)$, we can define the reachable set $\mathcal{R}[t_0, t_1]$ and controllable set $\mathcal{C}[t_0, t_1]$

Reachable set (Controllable-from-the-origin):

$$\mathcal{R}[t_0, t_1] = \left\{ x_1 \in \mathbb{R}^n \,\middle|\, \exists u(.), x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\mathrm{d}\tau \right\}.$$

Controllable set (Controllable-to-the-origin):

$$\mathcal{C}[t_0, t_1] = \Big\{ x_0 \in \mathbb{R}^n \,\Big|\, \exists v(.) = -u(.), \quad x_0 = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\mathrm{d}\tau \Big\}.$$

Research question.

The reachable and controllable sets, as defined depend on the control input $u(.)$. There are infinite possibilities for control input $u(.)$. How can we obtain the controllable and reachable sets in a tractable manner? We answer this question using the reachability and controllability Gramians.

Definitions

- Reachability Gramian
- Controllability Gramian

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For a given $t_1 > t_0$ Reachability Gramian is

$$W_R(t, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)^\top \Phi(t_1, \tau)\mathrm{d}\tau.$$

For a given $t_1 > t_0$ Controllability Gramian is

$$W_C(t, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B(\tau)^\top \Phi(t_0, \tau)\mathrm{d}\tau.$$

Practical result: Reachable and Controllable Subspaces.

- Reachable subspace

  $$\mathcal{R}[t_0, t_1] = \mathrm{Im} W_R(t_0, t_1).$$

  Moreover, if $x_1 = W_R(t_0, t_1)\eta_1 \in \mathrm{Im} W_R(t_0, t_1)$, then

  $$u(t) = B(t)^\top \Phi(t_1, t)^\top \eta_1, \quad t \in [t_0, t_1] \tag{3}$$

  can be used to transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$. This controller is the minimum-energy controller among all the controllers that can take the system $x(t_0) = 0$ to $x(t_1) = x_1$. Note that controller (3) is an open-loop controller.

- Controllable subspace

  $$\mathcal{C}[t_0, t_1] = \mathrm{Im} W_C(t_0, t_1).$$

  Moreover, if $x_0 = W_C(t_0, t_1)\eta_0 \in \mathrm{Im} W_C(t_0, t_1)$, then

  $$u(t) = -B(t)^\top \Phi(t_0, t)^\top \eta_0, \quad t \in [t_0, t_1] \tag{4}$$

  can be used to transfer the state from $x(t_0) = x_0$ to $x(t_1) = 0$. This controller is the minimum-energy controller among all the controllers that can take the system $x(t_0) = x_0$ to $x(t_1) = 0$. Note that controller (4) is an open-loop controller.

Take away result: Reachable and controllable system

Given any $t_1 > t_0$, system (1) is said to be

- Reachable, i.e., $\mathcal{R}[t_0, t_1] = \mathbb{R}^n$ if and only if $\mathrm{Im} W_R(t_0, t_1) = \mathbb{R}^n$, or $\mathrm{Rank}\, W_R(t_0, t_1) = n$.
- Controllable, i.e., $\mathcal{C}[t_0, t_1] = \mathbb{R}^n$ if and only if $\mathrm{Im} W_C(t_0, t_1) = \mathbb{R}^n$, or $\mathrm{Rank}\, W_C(t_0, t_1) = n$.

Example.

For system

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & t \end{bmatrix} x + \begin{bmatrix} \sqrt{t} \\ \sqrt{t}, \end{bmatrix}, \text{ where } \Phi(t, t_0) = \begin{bmatrix} 1 & -1 + \mathrm{e}^{\frac{t^2 - t_0^2}{2}} \\ 0 & \mathrm{e}^{\frac{t^2 - t_0^2}{2}} \end{bmatrix}$$

we have

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B(\tau)^\top \Phi(t_1, \tau) \mathrm{d}\tau =$$

$$\frac{1}{2}(-1+\mathrm{e}^{t_1^2 - t_0^2}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow R[t_0, t_1] = \mathrm{Im} W_R(t_0, t_1) = \text{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, this system is not fully reachable.

Problem setting: special case of LTI systems.

Now, we want to consider the special case of LTI systems

$$\dot{x} = Ax + Bu. \tag{5}$$

For this system we have

$$\Phi(t,\tau) = \mathrm{e}^{A(t-\tau)}.$$

Invoking the properties of matrix exponential, we can simplify or arrive at alternative ways to compute the controllable and reachable subspaces and tools to check for controllability and reachability of the system.

Main result for LTI systems.

Let

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots, A^{n-1}B \end{bmatrix}. \tag{6}$$

For any $t_0 > t_1$ we have

$$\mathcal{R}[t_0,t_1] = \operatorname{Im} W_R(t_0,t_1) = \operatorname{Im}\mathcal{C} = \operatorname{Im} W_C(t_0,t_1) = \mathcal{C}[t_0,t_1].$$

This result states that the controllable and reachable subspaces for LTI systems ate identical. Therefore, for LTI systems, most often when we talk about the concept of reachability/controllability under the topic of controllability and define the controllability as follows.

Because initialization does not matter for LTI systems we will start the system from $t_0 = 0$. For notational simplicity, for the finite time of interest we will use $t_1 = T$.

Definition: Controllablity for LTI systems.

The LTI system (5) or the pair (A,B) is said to be <u>controllable</u> if for any initial state $x_0 \in \mathbb{R}^n$ and any final state $x_1 \in \mathbb{R}^n$, there exists an input that transfers x_0 to x_1 in a <u>finite time</u>. Otherwise, (A,B) is said to be <u>uncontrollable</u>.

Definition: Controllablity Gramian for LTI systems.

For LTI system the controllability Gramian for any $t > 0$ is

$$W_C(t) = \int_0^t \mathrm{e}^{A\tau} B B^\top \mathrm{e}^{A^\top \tau} d\tau. \tag{7}$$

Take away result: Controllablility test for LTI systems

- The pair (A,B) is controllable if and only if for any finite $t > 0$, the controllability Gramian (7) is nonsigular, i.e., $\operatorname{Rank} W_C(t) = n$.
- Alternatively, (A,B) is controllable if and only if $\mathcal{C}$ is full rank, i.e.,

$$\operatorname{Rank} \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} = n \tag{8}$$

Minimum-energy controller for steering LTI systems .

If (A, B) is controllable the minimum-energy control for steering the system from $x(0) = x_0 \in \mathbb{R}^n$ to another point $x(T) = x_1 \in \mathbb{R}^n$ in the space is

$$u(t) = -B^\top e^{A^\top (T-t)} W_C^{-1}(T)(e^{AT}x_0 - x_1), \quad t \in [0, T]. \quad (9)$$

Note that this controller is open-loop.

Some facts about LTI systems.

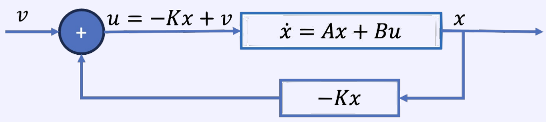
Controllability is invariant under similarity transformation

Let (A, B) and $(\bar{A}, \bar{B})$ be algebraically equivalent, i.e., $\exists T$ such that $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$.

$$(A, B) \text{ Controllable } \Leftrightarrow (\bar{A}, \bar{B}) \text{ Controllable.}$$

$$\text{Rank } \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \text{Rank } \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}.$$

Controllability is invariant under state-feedback



Here, state feedback $-Kx$ is used to stabilize the system and control the internal stability of the closed-loop system. Additional input v can be used to induce other behaviors such as reference tracking.

$$\dot{x} = Ax + Bu, \text{ let } u = -Kx + v \Rightarrow \dot{x} = \underbrace{(A - BK)}_{A_{cl}} x + Bv.$$

Q: Is (A_{cl}, B) is controllable? A: Yes.

$$(A, B) \text{ Controllable } \Leftrightarrow (A - BK, B) \text{ Controllable.}$$

Controllability is invariant under eigenvalue shift

Recall that $\text{eig}(A + \alpha I) = \text{eig}(A)$ for any $\alpha \in \mathbb{R}$.

$$(A, B) \text{ Controllable } \Leftrightarrow (A + \alpha I, B) \text{ Controllable.}$$

Popov-Belevitch-Hautus (PBH) test for controllability .

PBH Eigenvalue test

- (A, B) is controllable if and only if $\text{Rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ for any $\lambda \in \mathbb{C}$.

Because $\text{Rank}(A - \lambda I) = n$ for any $\lambda \notin \text{eig}(A)$, the eigenvalue test can also be stated as

- (A, B) is controllable if and only if $\text{Rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ for any $\lambda \in \text{eig}(A)$.

PBH Eigenvector test

- (A, B) is controllable if and only if for all the left eigenvector v of A we have $B^\top v \neq 0$.

Recall that $v \neq 0$ is a left eigenvector of A if $v^* A = \lambda v^*$. Every left eigenvector of A is the right eigenvector of $A^\top$. Therefore, the eigenvector test can also be stated as

- (A, B) is controllable if and only if for all the (right) eigenvector v of $A^\top$ we have $B^\top v \neq 0$.

Problem definition.

Regulation Problem (Stabilization Problem)
Starting from nonzero initial condition, find a controller that forces the states $x \to 0$ as $t \to \infty$.

Some observations.

For the LTI system (5), starting from any initial condition $x(0) = x_0 \in \mathbb{R}$, we know the trajectory of the system are given by

$$x(t) = \mathrm{e}^{A\,t} x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} B\, u(\tau)\, \mathrm{d}\tau. \qquad (10)$$

When $u = 0$ for all $t \geq 0$, the trajectory of the system is given by $x(t) = \mathrm{e}^{At} x_0$.

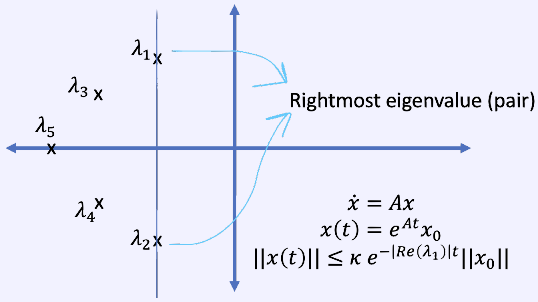
- When all the eigenvalues of the system have negative real part, then $\mathrm{e}^{At} \to 0$ as $t \to \infty$. Therefore $u = 0$ for $t \geq 0$ solves the regulation problem.

We usually do not want to only drive the states of the system to zero. We also want to enforce some performance factors on the system

- how fast we want to converge to zero
- we want certain transient response
- minimum energy

Here we are focused on the first two performance factor.

Rate of convergence of an LTI system $\dot{x} = Ax$ is no worst than the absolute value of the real part of the rightmost eigenvalue of A. That is, the rate of convergence of the internal dynamics is determined by the absolute value of the real part of the right most eigenvalue of system matrix A.



Research questions.

- Why do we still want to use a controller to regulate a Hurwitz system?
- How to design a regulator controller that can impose performance metrics such as how fast the system converges and control over transient response?

Solving Regulation using state feedback $u=-Kx$.

We want to solve the regulation problem using full state feedback $u=-Kx$. Using state feedback we obtain

$$\dot{x} = Ax + Bu \Rightarrow \dot{x} = \underbrace{(A-BK)}_{A_{cl}}x. \tag{11}$$

To impose our performance metrics we want control the trajectories of the closed-loop $\dot{x}=A_{cl}x$. The trajectories of this system are governed by eigenvalues of A_{cl}. So, to impose a certain response we need to use the feedback gain K to place the eigenvalues of A_{cl} at the desired locations that correlate with our performance metrics.

Research questions.

- When does a regulating state feedback exists?
- Can we have full control over eigenvalue placement of A_{cl} using state feedback gain K?
- If the answer to the previous question is yes, how can we obtain such K?

'Pole-placement' using state feedback $u=-K\,x$

Note: the problem of placing the eigenvalue of a LTI system often is called pole-placement!.

Theorem: Eigenvalue-placement to enforce a prespecified rate of convergence
Let (A,B) be controllable. For every $\alpha>0$, there always exists a K such that places all the eigenvalues of the closed-loop matrix $A_{cl}=A-BK$ on the complex semi plain $\mathrm{Re}(\lambda(A_{cl})\leq-\alpha$.
To find K, let $\mu\geq\alpha$ be such that $-A-\mu I$ is Hurwitz. Then,

$$K=\frac{1}{2}B^{\top}W^{-1}$$

results in $\mathrm{Re}(\lambda(A_{cl})\leq-\mu\leq-\alpha$ where

$$W=\frac{1}{2}\int_0^{\infty}\mathrm{e}^{(-\mu I-A)\tau}BB^{\top}\mathrm{e}^{(-\mu I-A)^{\top}\tau}\mathrm{d}\tau.$$

Theorem: Fully Controllable Eigenvalue-placement
Let (A,B) be controllable. Given any symmetric set of n complex numbers $\{\nu_1,\cdots,\nu_n\}$, there exists a full-state feedback matrix K such that the closed-loop system matrix $A-BK$ has eigenvalues equal to the given ν_i's.

Some definitions.

characteristic equation of a system $\dot{x}=Ax$ is

$$\Delta(A)=\det(\lambda I-A).$$

Given any symmetric set of n complex numbers $\{\nu_1,\cdots,\nu_n\}$ as the desired location for eigenvalues of $A-BK$, the desired characteristic equation of the closed-loop system is

$$\Delta(A-BK)=(\lambda-\nu_1)(\lambda-\nu_2)\cdots(\lambda-\nu_n)$$

Some definitions.

Controllable Canonical form
Consider $\dot{x} = Ax + Bu$ where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$. We say (A, B) is controllable canonical form when

$$A = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \cdots & -\alpha_{n-1} I_p & -\alpha_n I_p \\ I_p & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_p & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_p & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & \vdots & \ddots & & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_p & 0_{p\times p} \end{bmatrix}, \quad B = \begin{bmatrix} I_p \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \\ 0_{p\times p} \end{bmatrix} \tag{12}$$

where $\Delta(A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$.

Observation.

Any LTI system that is represented in controllable canonical form (12) is controllable.

Pole-placement for single input LTI systems: some preliminaries.

Every controllable single input LTI can be represented in controllable canonical form:
Consider the single input system (A, b). If (A, b) is controllable, there exists a similarity transformation matrix T such that $(\bar{A}, \bar{b}) = (T^{-1}AT, T^{-1}b)$ is in controllable canonical form.

Such transformation matrix T is given by $T = \mathcal{C}\bar{\mathcal{C}}^{-1}$ where $\mathcal{C}$ and $\bar{\mathcal{C}}$ are controllability matrices of (A, b) and $(\bar{A}, \bar{b})$, respectively. Note that

$$\bar{\mathcal{C}}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} \\ 0 & 0 & 1 & \alpha_1 & \cdots & \alpha_{n-3} \\ \vdots & \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $\{\alpha_1, \cdots \alpha_n\}$ are coefficients of the characteristic equation of (A, b):

$$\Delta(A) = \det(\lambda I - A) = \lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + \alpha_{n-1}\lambda + \alpha_n.$$

Recall that $\Delta(A) = \Delta(\bar{A})$ (properties of algebraically similar systems.)

Feedback gains that give same eigenvalues for $(A - bK)$ and $(\bar{A} - \bar{b}\bar{K})$
Let $\bar{K}$ be the feedback gain that places the eigenvalues of $(\bar{A} - \bar{b}\bar{K})$ at desired location $\{\nu_1, \cdots, \nu_n\}$. The gain that places the eigenvalues of $(A - bK)$ at the same desired location is

$$K = \bar{K}T^{-1}.$$

Why? A:

$$\begin{aligned} \{\nu_1, \cdots, \nu_n\} &= \text{eig}(\bar{A} - \bar{b}\bar{K}) = \text{eig}(\underbrace{T^{-1}AT}_{\bar{A}} - \underbrace{T^{-1}b}_{\bar{b}}\underbrace{KT}_{\bar{K}}) \\ &= \text{eig}(T^{-1}(A - bK)T) = \text{eig}(A - bK). \end{aligned}$$

Pole-placement for single input LTI systems: the procedures.

1. Find the coefficients $\alpha_1, \cdots, \alpha_n$ of the characteristic equation of $\Delta(A)$.
2. Form the desired characteristic equation

$$\begin{aligned}\Delta(A - bK) &= (\lambda - \nu_1)(\lambda - \nu_2)\cdots(\lambda - \nu_n)\\ &= \lambda^n + \hat{\alpha}_1\lambda^{n-1} + \hat{\alpha}_2\lambda^{n-2} + \cdots + \hat{\alpha}_{n-1}\lambda + \hat{\alpha}_n,\end{aligned}$$

   to obtain $\{\hat{\alpha}_1, \cdots, \hat{\alpha}_n\}$. Compute the elements of the desired state feedback gain $\bar{K} = \begin{bmatrix}\bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n\end{bmatrix}$ of the controllable canonical form from

$$\bar{k}_1 = \hat{\alpha}_1 - \alpha_1, \; \bar{k}_2 = \hat{\alpha}_2 - \alpha_2, \; \cdots, \; \bar{k}_n = \hat{\alpha}_n - \alpha_n$$

3. Find the desired feedback gain for $A - bK$ from $K = \bar{K}T^{-1}$, where T is the transformation matrix to obtain the controllable canonical form of (A, b).

Pole-placement for multi-input LTI system.

Main Theorem
Suppose the multi-input system (A, B) is controllable; $B \in \mathbb{R}^{n\times p}$. Then $(A - BF, Bv)$ is controllable for almost all $F \in \mathbb{R}^{p\times n}$ and $v \in \mathbb{R}^{p\times 1}$.

Some observations

- Notice that $Bv \in \mathbb{R}^{n\times 1}$. Therefore, $(A - BF, Bv)$ can be regarded as a model of a fictitious single input LTI system.
- The aforementioned theorem is guaranteeing that except for some singular cases, most often if your original system (A, B) is controllable, the fictitious system

$$(A - BF, Bv)$$

  we construct will be controllable as well.

Design procedure

1. Let the desired location of eigenvalues of $A - BK$ be $\text{eig}(A - BK) = \{\nu_1, \cdots, \nu_n\}$.
2. Choose $F \in \mathbb{R}^{p\times n}$ and $v \in \mathbb{R}^{p\times 1}$ to form the fictitious system (A_f, b_f) where $A_f = A - BF$ and $b_f = Bv$
3. Using the method you learned for pole-placement for single input systems, design $\bar{K} \in \mathbb{R}^{1\times p}$ such that

$$\text{eig}(A_f, b_f) = \{\nu_1, \cdots, \nu_n\}.$$

4. Noting that

$$A_f - b_f\bar{K} = A - BF - Bv\bar{K} = A - B\underbrace{(F + v\bar{K})}_{K} = A - BK,$$

   the desired gain to place the eigenvalues of (A, B) is $K = F + v\bar{K}$.

Research question: uncontrollable systems

What if the system is not controllable? Can we still stabilize this system using state feedback?

The case of uncontrollable LTI systems.

The LTI system (A, B) is uncontrollable when

$$\text{Rank}\,\underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} = m < n.$$

Our focus and the results derived next are to pave the way for finding addressing

Can we design a state feedback $u = -Kx$ to stabilize the system?

If yes, how can we design such controller?

Controllable decomposition.

Suppose $\text{Rank}\,\mathcal{C} = m < n$ for system (A, B). Then there exists a similarity transformation matrix T which results in $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$ with the following structure

$$\bar{A} = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline 0 & A_u \end{array}\right], \quad \bar{B} = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right], \tag{13}$$

where $A_c \in \mathbb{R}^{m\times m}$, $A_{12} \in \mathbb{R}^{m\times(n-m)}$, $A_u \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_c \in \mathbb{R}^{m\times p}$.

The transformation matrix for controllable decomposition is

$$T = \Big[\underbrace{t_1, \ldots, t_m}_{\text{basis of Im}\,\mathcal{C}} \;\Big|\; \underbrace{t_{m+1}, \ldots, t_n}_{\text{make } T \text{ invertible}}\Big]. \tag{14}$$

You can obtain the basis of $\text{Im}\,\mathcal{C}$ are m linearly independent columns of $\mathcal{C}$.

Take aways from Controllable decomposition.

Consider the controllable decomposition of (A, B). The pair (A_c, B_c) is controllable.

Recall that eigenvalues of two allegorically similar system matrix are the same. Therefore,

$$\text{eig}(A) = \text{eig}(\bar{A}) = \text{eig}(A_c) \cup \text{eig}(A_u). \tag{15}$$

Next note that if we use $\bar{K}$ to place the eigenvalues of $\bar{A}$ at some desired location $\{\nu_1, \cdots, \nu_n\}$ the gain

$$K = \bar{K}T^{-1} \tag{16}$$

is the gain places the eigenvalues of $A - BK$ at that location:

$$\{\nu_1, \cdots, \nu_n\} = \text{eig}(\bar{A} - \bar{B}\bar{K}) = \text{eig}(\underbrace{T^{-1}AT}_{\bar{A}} - \underbrace{T^{-1}B}_{\bar{B}}\underbrace{TK}_{\bar{K}})$$
$$= \text{eig}(T^{-1}(A - BK)T) = \text{eig}(A - BK).$$

Using the state feedback $u = -\underbrace{\begin{bmatrix}\bar{K}_c & \bar{K}_u\end{bmatrix}}_{\bar{K}}\bar{x}$, the closed-loop system in the transformed space is

$$\bar{A}_{cl} = \bar{A} - \bar{B}\bar{K} = \begin{bmatrix} A_c - B_c\bar{K}_c & A_{12} - B_cK_u \\ 0 & A_u \end{bmatrix}. \tag{17}$$

The eigenvalue of this closed-loop system are

$$\text{eig}(\bar{A}_{cl}) = \text{eig}(A_c - B_c\bar{K}_c) \cup \text{eig}(A_u). \tag{18}$$

Take aways from Controllable decomposition.

From (18) we can see that the state-feedback cannot move the eigenvalue of A_u. On the other hand, since (A_c, B_c) is controllable, we can use state feedback gain $\bar{K}_c$ to move the eigenvalues of A_c to wherever we want. As such, given (15).

- Eigenvalues of A_u are 'uncontrollable' eigenvalues of A
- Eigenvalues of A_c are 'controllable eigenvalues of A.

Number of uncontrollable eigenvalues.

When (A, B) is not controllable some of the eigenvalues of A are uncontrollable. The number of uncontrollable eigenvalues of A are equal to the $m = \text{Rank}\,\mathcal{C}$.

Research question.

From the observation above we can see when the system (A, B) is not controllable, we can still change the location of the eigenvalues but not all of them. We can change the location of the 'controllable' eigenvalues but we cannot change the location of the 'uncontrollable' eigenvalues. The question is how to identify the controllable and uncontrollable eigenvalues?

Identifying uncontrollable eigenvalues.

The m uncontrollable eigenvalues of uncontrollable system (A, B) can be obtained from

- constructing the controllable decomposition of the system to identify A_u. The uncontrollable eigenvalues are the eigenvalues of A_u.
- Use the PBH test. Any eigenvalue that fails the PBH eigenvalue test is an uncontrollable eigenvalue. You can also use the PBH eigenvector test. Any eigenvalue whose corresponding eigenvector fails the PBH test is an uncontrollable eigenvalue.

Pole-placement for stabilizable systems.

When (A, B) is not controllable but it is stabilizable, you can assign controllable eigenvalues to any other location using state feedback. The uncontrollable eigenvalues, no matter what state feedback gain you use, do not change their location, i.e., $\text{eig}(A_u) \subset \text{eig}(A - BK)$ for any K.

Suppose $\text{Rank}\,\mathcal{C} = m < n$ for (A, B), which means A has $n - m$ uncontrollable eigenvalues. Denote the eigenvalues of A with $\text{eig}(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_m, \lambda_{m+1}, \cdots, \lambda_n\}$, where $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ are controllable eigenvalues and $\{\lambda_{m+1}, \cdots, \lambda_n\}$ are uncontrollable eigenvalues. Considering the controllable decomposition of the system, this means that $\text{eig}(A_c) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and $\text{eig}(A_u) = \{\lambda_{m+1}, \cdots, \lambda_n\}$. Suppose that all the uncontrollable eigenvalues have negative real parts (A_u is Hurwitz), which means that the system is stabilizable. Then, for this system, we can reassign $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ to any other desirable location using state feedback but we cannot move $\{\lambda_{m+1}, \cdots, \lambda_n\}$.

To place eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ at desired locations $\{\nu_1, \cdots, \nu_m\}$ we use the controllable decomposition (13) representation of the system. Recall (17). Because (A_c, B_c) is controllable, we can use $\bar{K}_c$ to place eigenvalues of $A_c - B_c\bar{K}_c$ at the desired locations $\{\nu_1, \cdots, \nu_m\}$ using techniques we learned to place eigenvalue of controllable systems. Because $\bar{K}_u$ does not play any role in stabilizing the system, we can set $\bar{K}_u = 0$, i.e., $\bar{K} = \begin{bmatrix}\bar{K}_c & \bar{K}_u\end{bmatrix} = \begin{bmatrix}\bar{K}_c & 0\end{bmatrix}$. The gain that places eigenvalues of $A - BK$ at $\{\nu_1, \cdots, \nu_m\} \cup \{\lambda_{m+1}, \cdots, \lambda_n\}$ then is obtained from $K = KT^{-1}$ (recall (16)), where T is defined in (14).

Stablizable LTI system

- An LTI system (A, B) is stabilizable if it has no uncontrollable eigenvalue (fully controllable) or all its uncontrollable eigenvalues have negative real part (A_u is Hurwitz).
- If an uncontrollable (A, B) is stablilizable, we can always find a K such that all the eigenvalues of $A - BK$ has negative real part. The controllable eigenvalues of A can be assigned to other places, but the uncontrollable eigenvalue of A cannot be moved.
- When an LTI system is controllable we can always find a controller to drive the states of the system to zero from any initial conditions in finite time. This controller is not a state feedback (recall the open-loop minimum energy controller (9)).
- When an uncontrollable LTI system is stabilizable, then its states can be driven to zero using a control input in infinite time. In this case, no control input goes to the uncontrollable part of the system, so the system converges to zero with the rate of convergence of $\dot{x}_u = A_u x_u$.

References.

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