# Optimization Methods
# Lecture 13

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Consult: pages 276-297 (section 3.1 and 3.2) and sections 3.3, 3.3.1, 3.3.3 from Ref[1]

# Necessary Conditions for Optimality: equality and inequality conditions

Lagrangian function $L : \mathbb{R}^{n+m} \mapsto \mathbb{R}$: $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{r} \mu_j g_j(x)$

**Proposition (Karush-Huhn-Tucker Necessary conditions)**

Let $x^\star$ be a local minimum of
$$x^\star = \underset{x \in \mathbb{R}^n}{\text{argmin}} \; f(x) \quad \text{s.t.}$$
$$h_1(x) = 0, \cdots, h_m(x) = 0$$
$$g_1(x) \leqslant 0, \cdots, g_r(x) \leqslant 0$$

where $f$, $h_i$ and $g_j$ are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Assume the $x^\star$ is regular. Then there exists unique Lagrange multiplier vectors $\lambda^\star = (\lambda_1^\star, \cdots, \lambda_m^\star)$ and $\mu^\star = (\mu_1^\star, \cdots, \mu_r^\star)$, s.t.

$$\nabla_x L(x^\star, \lambda^\star, \mu^\star) = 0$$
$$\mu_j^\star \geqslant 0, \quad j = 1, \cdots, r$$
$$\mu_j^\star = 0, \quad \forall j \notin \underbrace{A(x^\star)}_{\text{active constraint set}}.$$

If in addition $f$ $g$ and $h$ are twice continuously differentiable we have

$$y^\top \nabla_{xx} L(x^\star, \lambda^\star, \mu^\star) \, y \geqslant 0,$$

for all
$y \in V(x^\star) = \{y \in \mathbb{R}^n | h_i(x^\star)^\top y = 0, \quad \forall i = 1, \cdots, m, \quad \nabla g_j(x^\star)^\top y = 0, \quad j \in A(x^\star)\}.$

# Solution approach

One approach for using necessary conditions to solve inequality constrained problems is to consider separately all the possible combinations of constraints being active or inactive.

# Constrained optimization: numerical example

$$\text{minimize } f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \text{ subject to}$$
$$g_1(x) = x_1^2 + x_2^2 - 5 \leqslant 0$$
$$g_2(x) = 3x_1 + x_2 - 6 \leqslant 0$$

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$$\nabla_x f(x) = \begin{bmatrix} 4x_1 + 2x_2 - 10 \\ 2x_1 + 2x_2 - 10 \end{bmatrix}, \quad \nabla_x g_1(x) = \begin{bmatrix} 2x_1 \\ 3 \end{bmatrix}, \quad \nabla_x g_2(x) = \begin{bmatrix} 2x_2 \\ 1 \end{bmatrix}$$

- H1: both constraints are inactive: $g_1 < 0$, $g_2 < 0$ and $\mu_1 = \mu_2 = 0$.
  FONC:

$$\left.\begin{array}{l} \nabla_{x_1} f(x) = 4x_1 + 2x_2 - 10 = 0 \\ \nabla_{x_2} f(x) = 2x_1 + 2x_2 - 10 = 0 \end{array}\right\} \Rightarrow \; x_1 = 0, x_2 = 5$$

  $g_1(x_1 = 0, x_2 = 5) = 20 > 0$ and $g_2(x_1 = 0, x_2 = -1 < 0$. Since H1 is not correct, this case is not possible.

- H2: both constraints are active: $g_1 = 0$, $g_2 = 0$ and $\mu_1, \mu_2 \geqslant 0$.

$$L(x, \mu) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 + \mu_1(x_1^2 + x_2^2 - 5) + \mu_2(3x_1 + x_2 - 6)$$

  FONC:

$$\left.\begin{array}{l} \nabla_{x_1} L(x, \mu) = 4x_1 + 2x_2 - 10 + 2\mu_1 x_1 + 3\mu_2 = 0 \\ \nabla_{x_2} L(x, \mu) = 2x_1 + 2x_2 - 10 + 2\mu_2 x_2 + \mu_2 = 0 \\ \nabla_{\mu_1} L(x, \mu) = x_1^2 + x_2^2 - 5 = 0 \\ \nabla_{\mu_1} L(x, \mu) = 3x_1 + x_2 - 6 = 0 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{ll} x = \begin{bmatrix} 2.1742 \\ -0.5225 \end{bmatrix}, \mu = \begin{bmatrix} -2.37 \\ 4.22 \end{bmatrix} & \text{since } \mu_1 < 0 \text{ this solution is not acceptable.} \\ x = \begin{bmatrix} 1.4258 \\ 1.7228 \end{bmatrix}, \mu = \begin{bmatrix} 1.37 \\ -1.02 \end{bmatrix} & \text{since } \mu_2 < 0 \text{ this solution is not acceptable.} \end{array}\right.$$

## Constrained optimization: numerical example

- H3: $g_1$ is inactive ($g_1 < 0,\ \mu_1 = 0$), and $g_2$ is active ($\mu_2 \geqslant 0$).

$$L(x,\mu) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 + \mu_2(3x_1 + x_2 - 6)$$

FONC:

$$\left.\begin{aligned} \nabla_{x_1} L(x,\mu) &= 4x_1 + 2x_2 - 10 + 3\mu_2 = 0 \\ \nabla_{x_2} L(x,\mu) &= 2x_1 + 2x_2 - 10 + \mu_2 = 0 \\ \nabla_{\mu_1} L(x,\mu) &= 3x_1 + x_2 - 6 = 0 \end{aligned}\right\} \Rightarrow x = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix},\ \mu_2 = -0.4.$$

since $\mu_2 < 0$ this solution is not acceptable.

- H4: $g_2$ is inactive ($g_2 < 0,\ \mu_2 = 0$), and $g_1$ is inactive ($\mu_1 \geqslant 0$).

$$L(x,\mu) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 + \mu_1(x_1^2 + x_2^2 - 5)$$

FONC:

$$\left.\begin{aligned} \nabla_{x_1} L(x,\mu) &= 4x_1 + 2x_2 - 10 + 2\mu_1 x_1 = 0 \\ \nabla_{x_2} L(x,\mu) &= 2x_1 + 2x_2 - 10 + 2\mu_1 x_2 = 0 \\ \nabla_{\mu_1} L(x,\mu) &= x_1^2 + x_2^2 - 5 = 0 \end{aligned}\right\} \Rightarrow x^\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mu_1^\star = 1.$$

since $\mu_1 \geqslant 0$ this solution is qualified as KKT solution.

Now we need to validate H4: $g_2(x_1 = 1, x_2 = 2) = -1 < 0$, therefore H4 is correct.
SONC:

$$y \nabla_{xx} L(x^\star, \mu^\star) y \geqslant 0 \text{ for } y \in V(x^\star) = \{y \in \mathbb{R}^2 | \nabla g_1(x^\star)^\top y = 0\} = \{y \in \mathbb{R}^2 | \begin{bmatrix} 2 & 4 \end{bmatrix} y = 0\}$$

Since $\nabla_{xx} L(x^\star, \mu^\star) = \begin{bmatrix} 4 + 2\mu_1^\star & 2 \\ 2 & 2 + 2\mu_1^\star \end{bmatrix} > 0$ ($\mu^\star = 1$), then SONC condition is definitely satisfied. Also since the condition holds for strict $> 0$, then the second order sufficiency condition is satisfied and $x_1^\star = 1, x_2^\star = 2$ is a local minimizer.

# Fritz Jonh Necessary Conditions for Optimality

Lagrangian function $L : \mathbb{R}^{n+m} \mapsto \mathbb{R}$: $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{r} \mu_j g_j(x)$

**Proposition (Fritz Jonh Necessary Conditions for Optimality)**

Let $x^\star$ be a local minimum of

$$x^\star = \underset{x \in \mathbb{R}^n}{\text{argmin}} \ f(x) \quad \text{s.t.}$$
$$h_1(x) = 0, \cdots, h_m(x) = 0$$
$$g_1(x) \leqslant 0, \cdots, g_r(x) \leqslant 0$$

where $f$, $h_i$ and $g_j$ are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Then there exist a scalar $\mu_0^\star$ and Lagrange multiplier vectors $\lambda^\star = (\lambda_1^\star, \cdots, \lambda_m^\star)$ and $\mu^\star = (\mu_1^\star, \cdots, \mu_r^\star)$, s.t.

- **(i)** $\mu_0^\star \nabla_x f(x^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla_x h_i(x^\star) + \sum_{i=1}^{m} \mu_j^\star \nabla_x g_j(x^\star) = 0$
- **(ii)** $\mu_j^\star \geqslant 0, \quad j = 1, \cdots, r$
- **(iii)** $\lambda_1^\star, \cdots, \lambda_m^\star$, $\mu_1^\star, \cdots, \mu_r^\star$ and $\mu_0^\star$ are not all equal to zero
- **(iv)** In every neighborhood $N$ of $x^\star$ there is an $x \in N$ such that $\lambda_i^\star h_i(x) > 0$ for all $i$ with $\lambda_i^\star \neq 0$ and $\mu_j^\star g_j(x) > 0$ for all $j$ with $\mu_j^\star$.

Fritz Jonh Necessary Conditions for Optimality does not require that $x^\star$ be regular.

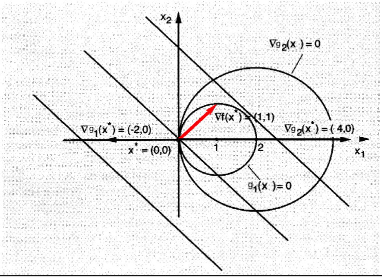
# Numerical example: use of Fritz Jonh condition

- **Regular point of a set of constraints**: A feasible vector $x$ for which the constraint gradients $\{\nabla h_1(x), \cdots, \nabla h_m(x)\}$ are linearly independent.
- For a local minimum that is not regular, the KKT condition does not apply

$$\text{minimize } f(x) = x_1 + x_2, \quad s.t.$$

$$g_1(x) = (x_1 - 1)^2 + x_2^2 - 1 \leqslant 0, \quad g_2(x) = -(x_1 - 2)^2 - x_2^2 + 4 \leqslant 0.$$

- $x^\star$ is not regular. Therefore, this problem cannot be solved using Lagrange multiplier theorem (KKT condition).
- $\nabla f(x^\star)$ cannot be written as linear combination of $\nabla g_1(x^\star)$ and $\nabla g_2(x^\star)$

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Fritz Jonh Necessary Conditions for Optimality:

$$\mu_0 \nabla_x f(x) + \mu_1 \nabla g_1(x) + \mu_2 \nabla h_2(x) = 0 \Rightarrow \begin{cases} \mu_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 2(x_1 - 1) \\ 2x_2 \end{bmatrix} + \mu_2 \begin{bmatrix} -2(x_1 - 2) \\ -2x_2 \end{bmatrix} = 0 \\ (x_1 - 1)^2 + x_2^2 - 1 = 0 \\ -(x_1 - 2)^2 - x_2^2 - 4 = 0 \end{cases}$$

$x_1^\star = 0, \quad x_2^\star = 0, \ \mu_0^\star = 0,$ for any $\mu_1^\star, \mu_2^\star \geqslant 0$ such that $\mu_1^\star = 2\mu_2^\star$ condition (i)-(iii) of Fritz Jonh necessary condition is satisfied. Since From the geometry of the problem, it can be verified that condition