Optimization Methods
Lecture 1

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Material to review: pages 1-15 of Ref[1] and Chapter 1 of Ref [2].

Examples of optimization problems:

- profit or loss in business setting
- speed, distance or fuel consumption in physical problems
- expected return in risky environment
- social welfare problems in the context of government planning

Optimization theory provides
a suitable framework for analyzing and providing solution

Mathematical models of optimization

Elements of optimization problem: A constraint set X and a cost function f that maps elements of X into a real numbers

- X: The set of constraints of the available decisions x
- $f(x)$: the cost is a scalar measure of undesirability of choosing decision x

Objective: find an optimal decision $x^\star \in X$ such that $f(x^\star) \leqslant f(x)$ for $\forall x \in X$

$$\text{minimize } f(x) \quad s.t. \quad x \in X \qquad \text{or} \qquad x^\star = \text{argmin } f(x) \quad s.t. \quad x \in X$$

In our studies $X \subseteq R^n$ (i.e., $x \in \mathbb{R}^n$)

Mathematical models of optimization

Different types of optimization: depends on $f(x)$ and X

- **Continuous vs. Discrete**
  - Continuous problems: Constraint set X is infinite and has "continuous" character

    Examples: $X = \mathbb{R}^n$

    $$X = \{x \in \mathbb{R}^2 \,|\, x_2 \geqslant x_1^2, \quad x_1 + x_2 \leqslant 2\}$$

    Tools to analyze: Mathematics of calculus and convexity

  - Discrete problems: mostly because constraint set X is finite

    Examples: $\begin{cases} \text{routing} \\ \text{scheduling} \\ \text{Matching} \end{cases}$

    Important class of discrete problems: integer programing (decision value from some range of integer numbers such as $\{0, 1\}$)

    Tools to analyze: Combinational and discrete mathematics; Other special methodology that relate to continuous problems

Mathematical models of optimization

Different types of optimization: depends on $f(x)$ and X

- **Nonlinear programming**
  - cost f is nonlinear and/or
  - X is specified by nonlinear equations and inequalities
- **Linear programming**
  - cost f is linear
  - X is specified by linear inequality constraints

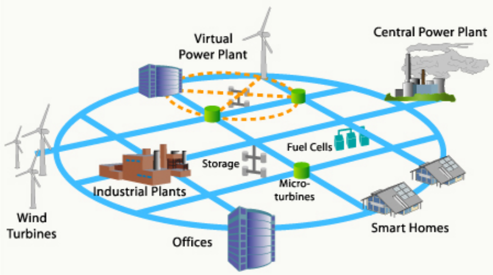
Our focus: nonlinear programing for continuous optimization problems.

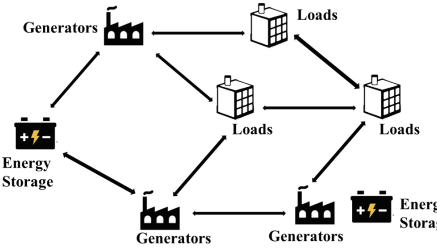
$$
\begin{aligned}
x^\star = & \operatorname*{argmin}_{x \in \mathbb{R}^n} f(x) \quad s.t. \\
& h_i(x) = 0, \quad i \in \{1, \cdots, q\} \\
& g_i(x) \geqslant 0, \quad i \in \{1, \cdots, p\}
\end{aligned}
$$

f,h,g: continuously differentiable function of x
e.g., $f \in C^1$ continuously differentiable
e.g., $f \in C^2$ both f and its first derivative are continuously differentiable

- Economic Dispatch

$$\begin{aligned} \text{minimize} \quad & f(p) = f^1(p^1) + \cdots + f^N(p^N) \\ \text{subject to} \quad & p_{min_i} \leqslant p^i \leqslant p_{max_i} \quad i \in \{1, \cdots, N\} \\ & p^1 + \cdots + p^N = Demand \end{aligned}$$





- Economic dispatch with storage and transmission loss

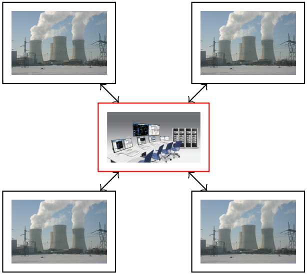
Continuous Optimization: Examples

- Economic Dispatch

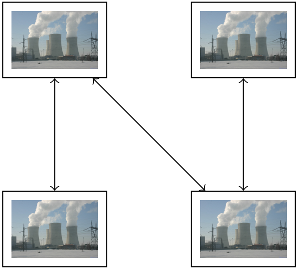
$$\text{minimize} \quad f(p) = f^1(p^1) + \cdots + f^N(p^N)$$

$$\text{subject to} \quad p_{min_i} \leqslant p^i \leqslant p_{max_i} \quad i \in \{1, \cdots, N\}$$

$$p^1 + \cdots + p^N = Demand$$



Central Operation



Distributed Operation

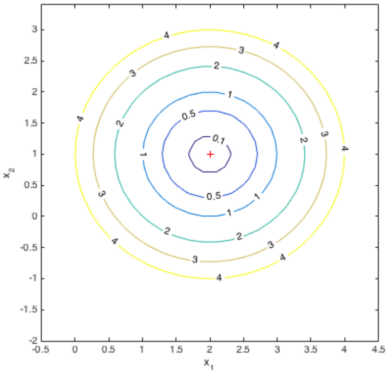
Continuous Optimization: Examples

Unconstraint optimization

$$x^\star = \underset{x \in \mathbb{R}^n}{\text{argmin}}\ f(x)$$

Example:

$$x^\star = \underset{x \in \mathbb{R}^2}{\text{argmin}}\ \underbrace{(x_1 - 2)^2 + (x_2 - 1)^2}_{f(x)}$$




```
x1 = -0.5:0.2:4.5;
x2 = -2:0.2:3.5;
[X1,X2] = meshgrid(x1,x2);
Z = (X1-2).^2+(X2-1).^2;
v = [0.1,0.1,0.5,0.5,1,1,2,2,3,3,4,4];
figure
contour(X1,X2,Z,v,'ShowText','on')
hold on
plot(2,1,'r+')
pbaspect([1 1 1])
xlabel('x_1')
ylabel('x_2')
```

Example:

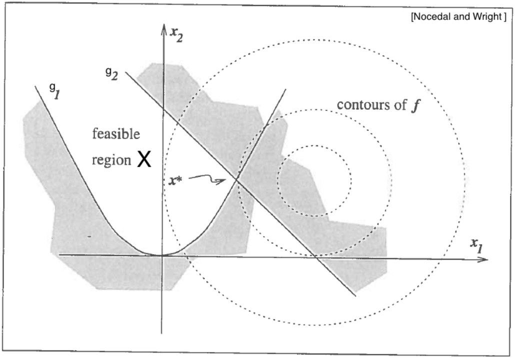
$$x^\star = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \; f(x) \quad \text{s.t.}$$

$$h_i(x) = 0, \quad i \in \{1, \cdots, q\}$$

$$g_i(x) \geqslant 0, \quad i \in \{1, \cdots, p\}$$

$$x^\star = \underset{x \in \mathbb{R}^2}{\operatorname{argmin}} \; \underbrace{(x_1 - 2)^2 + (x_2 - 1)^2}_{f(x)} \quad \text{s.t.}$$

$$g_i : \begin{cases} -x_1^2 + x_2 \geqslant 0 \\ -x_1 - x_2 + 2 \geqslant 0 \end{cases}$$



- $x^\star \in \mathbb{R}^n$ is an **unconstrained local minimum of** f if

$$\exists\, \epsilon > 0 \;\; \text{s.t.} \;\; f(x^\star) \leqslant f(x), \qquad \forall x \text{ with } \|x - x^\star\| < \epsilon,$$

- $x^\star \in \mathbb{R}^n$ is an **unconstrained global minimum of** f if

$$f(x^\star) \leqslant f(x), \qquad \forall x \in \mathbb{R}^n,$$

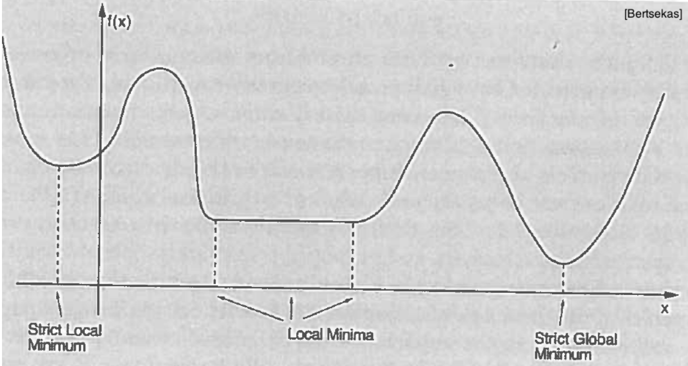
- $x^\star \in \mathbb{R}^n$ is an **unconstrained strict local minimum of** f if

$$\exists\, \epsilon > 0 \;\; \text{s.t.} \;\; f(x^\star) < f(x), \qquad \forall x \text{ with } \|x - x^\star\| < \epsilon,$$

- $x^\star \in \mathbb{R}^n$ is an **unconstrained strict global minimum of** f if

$$f(x^\star) < f(x), \qquad \forall x \in \mathbb{R}^n,$$

Local and global minima of a constrained optimization problem



Local and global minima of an constrained optimization problem

Let X be the constraint set.

- $x^\star \in X$ is a **local minimum of** f if

$$\exists\, \epsilon > 0 \;\; s.t. \;\; f(x^\star) \leqslant f(x), \qquad \forall x \in X \text{ with } \|x - x^\star\| < \epsilon,$$

- $x^\star \in X$ is a **global minimum of** f if

$$f(x^\star) \leqslant f(x), \qquad \forall x \in X,$$

- $x^\star \in X$ is a **constrained strict local minimum of** f if

$$\exists\, \epsilon > 0 \;\; s.t. \;\; f(x^\star) < f(x), \qquad \forall x \in X \text{ with } \|x - x^\star\| < \epsilon,$$

- $x^\star \in X$ is a **constrained strict global minimum of** f if

$$f(x^\star) < f(x), \qquad \forall x \in X,$$

Some preliminaries

- Gradient of a $f(x) : \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Example:
$f(x) = \frac{1}{2}(x_1 - 2)^2 + x_1 x_2 x_3 + \frac{1}{3}(x_3 + 4)^3$

$$\nabla f(x) = \begin{bmatrix} (x_1 - 2) + x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 + (x_3 + 4)^2 \end{bmatrix}$$

- Hessian of a $f(x) : \mathbb{R}^n \to \mathbb{R}$, $f \in C^2$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} = [\frac{\partial^2 f}{\partial x_i \partial x_j}]_{ij}, \quad \text{it is a symmetric matrix}$$

Example: $f(x) = \frac{1}{2}(x_1 - 2)^2 + x_1 x_2 x_3 + \frac{1}{3}(x_3 + 4)^3$

$$\nabla^2 f(x) = \begin{bmatrix} 1 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 2(x_3 + 4) \end{bmatrix}$$

References

[1] Nonlinear Programming: 3rd Edition, by D. P. Bertsekas

[2] Linear and Nonlinear Programming, by D. G. Luenberger, Y. Ye

[3] Numerical Optimization, by J. Nocedal and S. J. Wright