HOMEWORK 2
MAE 206- OPTIMIZATION METHODS
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Problem 1. *This problem is a practice on computing the admissible range of a fixed stepsize for the steepest descent algorithm.* Consider the unconstrained optimization problem with cost function below, which we given in problem 4 of homework 1. For this problem find the valid range of fixed stepsize for the steepest descent algorithm.

$$f(x) = 2\,x_1^2 + x_1\,x_2 + x_2^2 + x_2\,x_3 + x_3^2 - 6\,x_1 - 7\,x_2 - 8\,x_3 + 9.$$

Use the range you obtained to justify the observation that you made about the convergence of the algorithm for constant stepsizes that were given in homework 1 (i.e., $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 1$).

Problem 2. *This problem is a practice on line search methods.*
Consider the unconstrained optimization problem

$$f(x) = \frac{x^2 + 3x}{x^2 + 3x + 5},$$

- Compute the stationary point(s) of the function from the first order necessary conditions
- Find a valid range of x that brackets the minimum.
- Confirm that the range $[-3, 1]$ brackets the minimizer $x^\star$ (assume that you do not know the value of $x^\star$). Starting from bracket $[-3, 1]$ preform 4 steps of the Golden Section and quadratic fit algorithms to find the minimizer (by hand or use a Matlab code). For the quadratic fit algorithm choose the third point you need to initialize the algorithm $x = 0$. Report your steps. Compare the size of your forth bracket for each of these algorithms

Problem 3. *This problem is a practice on line search methods.*
Consider the unconstrained optimization problem

$$f(x) = x^2 + 2x + 1,$$

- Compute the stationary point of the function from the first order necessary conditions
- Let $x_0 = 4$, preform 2 steps of steepest descent algorithm with
  (a) exact line search
  (b) Armijo backtracking line search with parameters ($\sigma = 0.1$, $\beta = 0.5$ and starting at $\alpha = 1.6$)
  (c) Armijo backtracking line search with parameters ($\sigma = 0.05$, $\beta = 0.1$ and starting at $\alpha = 1.6$).
  Compute and compare $E(x_2) = f(x_2) - f(x^\star)$ of the cases above.

– – – – – – – – – – – – **Sample Code** – – – – – – – – – – – – – –

Find the minimizer of

$$f(x) = \frac{1}{2}x^{\top}\begin{bmatrix}70 & 2\\ 2 & 2\end{bmatrix}x - \begin{bmatrix}1\\ 2\end{bmatrix}x$$

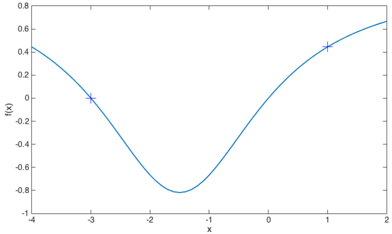
A simple code for steepest descent with exact line search for a quadratic cost function (see problem 3 of HW 1) :

```
close all
Q=[70 2
    2 2];
b=[1 2]';

x_s=inv(Q)*b; %exact value of the minimizer

plot(x_s(1),x_s(2),'g*')
hold on
x1 = -1:0.01:1.2;
x2 = -1:0.01:4;
[X1,X2] = meshgrid(x1,x2);
v=[0.1,0.1,0.5,0.5,1,1,5,5,10,10,15,15,20,20,25,25,30,30,40,40,50,50,60,60];
Z = 0.5*(Q(1,1)*X1.*X1+2*Q(1,2)*X1.*X2+Q(2,2)*X2.*X2)-b(1)*X1-b(2)*X2;
contour(X1,X2,Z,v,'ShowText','on')
x_sol(:,1)=[1 4]';
plot(x_sol(1,1),x_sol(2,1),'r+')
for k=1:5
    g(:,k)=Q*x_sol(:,k)-b;
    alpha(k)=(g(:,k)'*g(:,k))/(g(:,k)'*Q*g(:,k));
    x_sol(:,k+1)=x_sol(:,k)-alpha(k)*g(:,k);
    plot(x_sol(1,k+1),x_sol(2,k+1),'r+')
end
plot(x_sol(1,:),x_sol(2,:),'r')
```

Golden Section method

You can find $x^\star \in \mathbb{R}$ in $\quad x^\star = \underset{x \in \mathbb{R}}{\operatorname{argmin}} f(x) \quad$ using Golden section method.

Given $[x_k, \bar{x}_k]$, determine $[x_{k+1}, \bar{x}_{k+1}]$ such that $x^\star \in [x_{k+1}, \bar{x}_{k+1}]$.



- **Initialization**: $[x_0, \bar{x}_0]$ (in you HW problem $[x_0, \bar{x}_0] = [-3, 1]$)
- **Step k**: $\begin{cases} b_k = x_k + \tau(\bar{x}_k - x_k), \\ \bar{b}_k = \bar{x}_k - \tau(\bar{x}_k - x_k), \end{cases}$
- **compute** $f(b_k)$ and $f(\bar{b}_k)$

  (1) If $f(b_k) < f(\bar{b}_k)$: $\begin{cases} x_{k+1} = x_k, \quad \bar{x}_{k+1} = b_k & \text{if } f(x_k) \leqslant f(b_k) \\ x_{k+1} = x_k, \quad \bar{x}_{k+1} = \bar{b}_k & \text{if } f(x_k) > f(b_k) \end{cases}$

  (2) If $f(b_k) > f(\bar{b}_k)$: $\begin{cases} x_{k+1} = \bar{b}_k, \quad \bar{x}_{k+1} = \bar{x}_k & \text{if } f(\bar{b}_k) \geqslant f(\bar{x}_k) \\ x_{k+1} = b_k, \quad \bar{x}_{k+1} = \bar{x}_k & \text{if } f(\bar{b}_k) < f(\bar{x}_k) \end{cases}$

  (3) If $f(b_k) = f(\bar{b}_k)$: $\quad x_{k+1} = b_k, \quad \bar{x}_{k+1} = \bar{b}_k.$
- **Stop**: If $(\bar{x}_k - x_k) < \epsilon$

Strictly unimodal f: the interval $[x_k, \bar{x}_k]$ contains $x^\star$ and $(\bar{x}_k - x_k) \to 0$