Saturation-tolerant average consensus with controllable rates of convergence

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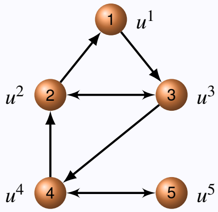
Problem definition

Static Average Consensus

- Autonomous and cooperative agents

$$\dot{x}^i = -c^i, \quad x^i, c^i \in \mathbb{R}$$

- x^i: agreement state
- c^i: driving command



- Design $c^i = f(i, \text{neighbors})$ s.t. $\forall i \in \{1, \ldots, N\}$

$$x^i(t) \to \frac{1}{N}\sum_{j=1}^{N} u^j, \quad t \to \infty$$

Applications: coordination and information fusion

- multi-robot coordination
- distributed optimization
- distributed fusion in sensor networks
- smart meters

Static average consensus is one of the most studied problems in networked systems

- Inspired by analysis of group behavior (flocking) in nature: Vicsek 95, Reynolds 87, Toner and Tu 98
- Mathematical models of static consensus and averaging: Jadbabaie et al. 03, Olfati Saber and Murray 03 and 04, Boyd et al. 05

Previous literature:

- Focus on convergence to consensus: time delay, switching, noisy links
- Focus on increase rate of convergence,
- No explicit attention to rate of convergence of individual agents
- No explicit attention to limited control authority

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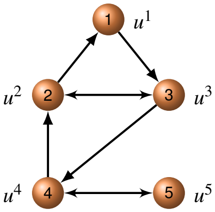
- x^i: Agreement state - c^i: Driving command



Design $c^i = f(i, \text{neighbors})$ s.t.

1. $x^i \to \frac{1}{N}\sum_{j=1}^{N} u^j, \;\; t \to \infty,$ **with rate β^i**
   - Agents with limited control authority opt for slower rate
   - Consistent response over different communication topologies
   - Control over time of arrival
2. $x^i \to \frac{1}{N}\sum_{j=1}^{N} u^j, \;\; t \to \infty,$ **even though $\dot{x}^i = -\mathrm{sat}_{\bar{c}^i}(c^i)$**
   - Average consensus is achieved despite limited control authority

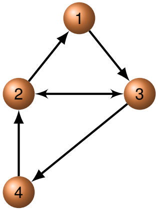
Network model

Communication topology: weighted digraph $\mathcal{G}(V, \mathcal{E}, A)$

- Node set: $V = \{1, \cdots, N\}$
- Edge set: $\mathcal{E} \subseteq V \times V$
- Weights (for $i, j \in \{1, \ldots, N\}$)

$$a_{ij} > 0 \textit{ if } (i,j) \in \mathcal{E}, \; a_{ij} = 0 \textit{ if } (i,j) \notin \mathcal{E}$$

- Strongly connected: $i \to j$ for any i, j



- Weight-balanced:

$$\sum_{j=1}^{N} a_{ji} = \sum_{j=1}^{N} a_{ij}, \quad i \in V$$

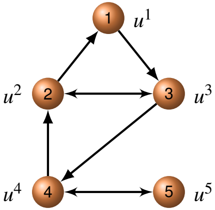
- Laplacian matrix: $L = D^{\text{out}} - A$

$$A : \text{Adjacency matrix}; \quad D : \text{out degree}, \; D_{ii}^{\text{out}} = \sum_{j=1}^{N} a_{ij}, \quad i \in V$$

Laplacian algorithm: a solution by R. Olfati-Saber and R. Murray 2003, 2004

$$\dot{x}^i = -c^i, \quad x^i, c^i \in \mathbb{R}$$

$$c^i = \sum_{j=1}^{N} a_{ij}(x^i - x^j), \quad x^i(0) = u^i$$



- Unbounded c^i
- Weight-balanced
- Strongly connected

- $x^i \to \frac{1}{N}\sum_{j=1}^{N} x^j(0) = \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$

- Exponential convergence with rate $\hat{\lambda}_2 = \min\{\lambda(\frac{1}{2}(L+L^\top)) > 0\}$

$$\left|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j\right| \leqslant \left|\boldsymbol{x}(t) - \frac{1}{N}\sum_{j=1}^{N} u^j \mathbf{1}_N\right| \leqslant \left|\boldsymbol{x}(0) - \frac{1}{N}\sum_{j=1}^{N} u^j \mathbf{1}_N\right| \mathrm{e}^{-\hat{\lambda}_2 t}, \quad t \geqslant 0$$

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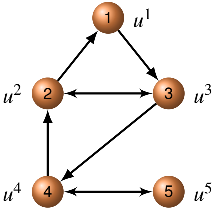

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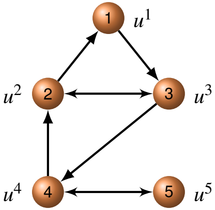

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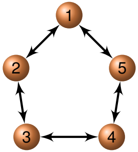
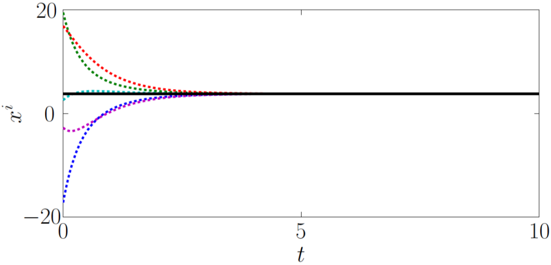
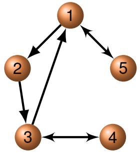
Response of Laplacian algorithm for two different graph topologies

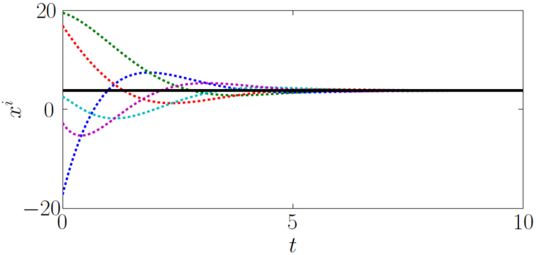


$\hat{\lambda}_2 = 1.38$





$\hat{\lambda}_2 = 0.5$



Think about physical processes

$$\dot{x}^i = -c^i$$

- Accommodate agents with limited control authority
- Consistent transient across all communication topologies
- Control over time of arrival

Every agent controls its own convergence rate

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Static average consensus: controllable rate of convergence at each agent

Problem Definition

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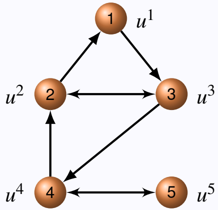
- x^i: Agreement state - c^i: Driving command

Design $c^i = f(i, \text{neighbors})$ s.t.

$$x^i \to \frac{1}{N}\sum_{j=1}^{N} u^j, \quad t \to \infty \text{ **with rate** } \beta^i, \text{ i.e.}$$

$$\left| x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j \right| \leqslant \kappa \left| x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j \right| \mathrm{e}^{-\beta^i t}$$



Design methodology

- Simplest dynamics: $x^i \to \frac{1}{N}\sum_{j=1}^{N} u^j$ **with rate** β^i

$$\dot{x}^i = -\beta^i(x^i - \frac{1}{N}\sum_{j=1}^{N} u^j)$$

- Requirement: **fast** dynamics to generate $\frac{1}{N}\sum_{j=1}^{N} u^j$ in a **distributed manner**!
- Two-time scales:
  - **Fast dynamics**: $\dot{z} = -Lz, \;\; z^i(0) = u^i : \quad z^i \to \frac{1}{N}\sum_{j=1}^{N} u^j$
  - **Slow dynamics**: $\dot{x}^i = -\beta^i(x^i - \frac{1}{N}\sum_{j=1}^{N} u^j)$

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Proposed solution

$$\begin{cases} \epsilon\, \dot{z}^i = \sum_{j=1}^N a_{ij}(z^i - z^j), & z^i(0) = u^i, \\ \dot{x}^i = -\beta^i(x^i - z^i), & x^i(0) \in \mathbb{R}, \end{cases} \qquad i \in \{1, \ldots, N\}$$

Lemma

For strongly connected and weight-balanced digraphs, $\forall\ \epsilon, \beta^i > 0$*,*

$$x^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j, \text{ as } t \to \infty, \quad i \in \{1, \ldots, N\},$$

exponentially fast, with a rate $\min\{\beta^i, \epsilon^{-1}\hat{\lambda}_2\}$*.*

$\hat{\lambda}_2 = \min\{\lambda(\frac{1}{2}(L + L^T)) > 0\}$

Static average consensus: controllable rate of convergence at each agent

Sketch of the proof:

$$\dot{z} = -\epsilon^{-1}\boldsymbol{L}z, \quad z^i(0) = u^i \in \mathbb{R},$$
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- Laplacian algorithm :

$$\left|z^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j\right| \leqslant \left|z(0) - (\frac{1}{N}\sum_{j=1}^{N} u^j)\mathbf{1}_N\right| \mathrm{e}^{-\epsilon^{-1}\hat{\lambda}_2 t}, \quad t \geqslant 0$$

- Solution of the agreement dynamics:

$$x^i(t) = x^i(0)\mathrm{e}^{-\beta^i t} + \beta^i \int_0^t \mathrm{e}^{-\beta^i(t-\tau)} z^i(\tau) d\tau$$

  - For $\beta^i = \epsilon^{-1}\hat{\lambda}_2$:

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| \leqslant |x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j|\mathrm{e}^{-\beta^i t} + t\,\beta^i \kappa_z \mathrm{e}^{-\beta^i t};$$

  - For $\beta^i \neq \epsilon^{-1}\hat{\lambda}_2$:

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| \leqslant \kappa_x \mathrm{e}^{-\beta^i t} + \frac{\beta^i \kappa_z}{\beta^i - \epsilon^{-1}\hat{\lambda}_2}(\mathrm{e}^{-\epsilon^{-1}\hat{\lambda}_2 t} - \mathrm{e}^{-\beta^i t}).$$

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$\hat{\lambda}_2 = \min\{\lambda(\frac{1}{2}(L + L^T)) > 0\}$

Problem Def.: A static average consensus algorithm with controllable rate of convergence at each agent

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Design $c^i = f(i, \text{neighbors})$ s.t.

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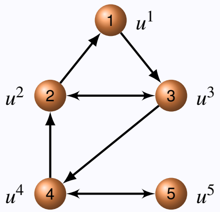

solution

$$\begin{cases} \epsilon\, \dot{z}^i = -\sum_{j=1}^{N} a_{ij}(z^i - z^j), & z^i(0) = u^i, \\ \dot{x}^i = -\beta^i(x^i - z^i), & x^i(0) \in \mathbb{R}, \end{cases} \qquad i \in \{1, \dots, N\}$$

Rate of convergence of x^i is $\min\{\beta^i, \epsilon^{-1}\hat{\lambda}_2\}$, then

$$\epsilon \leqslant \frac{\hat{\lambda}_2}{\bar{\beta}}, \quad \bar{\beta} = \max\{\beta^1, \cdots, \beta^N\}$$

$\hat{\lambda}_2 = \min\{\lambda(\frac{1}{2}(L + L^T)) > 0\}$

Extension to networks with noisy links, switching networks, time delays

An alternative proof of the convergence of the proposed algorithm:

$$\begin{aligned} \dot{z} &= -\epsilon^{-1} \boldsymbol{L} z, \quad z^i(0) = u^i \in \mathbb{R} \\ \dot{x}^i &= -\beta^i (x^i - z^i), \quad x^i(0) \in \mathbb{R} \end{aligned} \quad \xrightarrow[q^i = z^i - \frac{1}{N}\sum_{j=1}^N u^j]{p^i = x^i - \frac{1}{N}\sum_{j=1}^N u^j} \quad \begin{aligned} \dot{z} &= -\epsilon^{-1} \boldsymbol{L} z \\ \dot{p}^i &= -\beta^i (p^i - q^i) \end{aligned}$$

- Laplacian algorithm: $z^i \to \frac{1}{N}\sum_{j=1}^N u^j, \ (q^i \to 0)$, as $t \to \infty, \quad \forall i \in \{1, \ldots, N\}$
- $\dot{p}^i = -\beta^i p^i$ is exponentially stable
- $\dot{p}^i = -\beta^i (p^i - q^i)$ is a linear system with vanishing input

$$\therefore \ \boldsymbol{p^i \to 0, \ (x^i \to \frac{1}{N}\sum_{j=1}^N u^j)}, \text{ as } \boldsymbol{t \to \infty}, \quad \forall \boldsymbol{i} \in \{\boldsymbol{1}, \ldots, \boldsymbol{N}\}$$

Our proposed algorithm inherits any result related to noisy links, switching networks, time delays of the Laplacian algorithm

An alternative proof of the convergence of the proposed algorithm:

$$\dot{z} = -\epsilon^{-1}\boldsymbol{L}z,\ \ z^i(0) = u^i \in \mathbb{R} \qquad \xrightarrow[q^i = z^i - \frac{1}{N}\sum_{j=1}^N u^j]{p^i = x^i - \frac{1}{N}\sum_{j=1}^N u^j} \qquad \dot{z} = -\epsilon^{-1}\boldsymbol{L}z$$

$$\dot{x}^i = -\beta^i(x^i - z^i),\ \ x^i(0) \in \mathbb{R} \qquad\qquad \dot{p}^i = -\beta^i(p^i - q^i)$$

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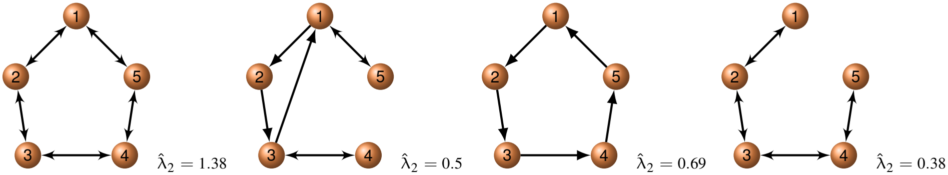
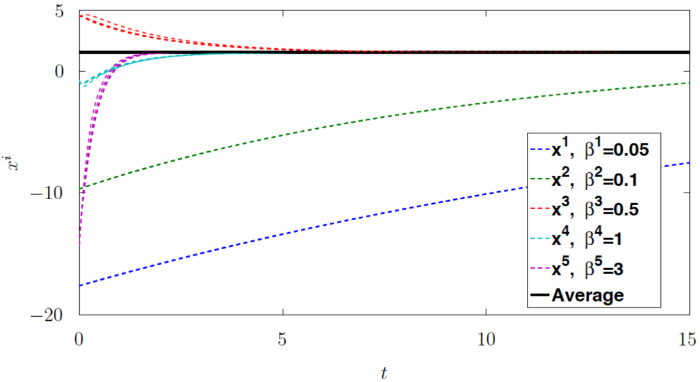
Our proposed algorithm inherits any result related to noisy links, switching networks, time delays of the Laplacian algorithm

The proposed static average consensus: example





Desired rates and consistent transient are imposed by using $\epsilon = 0.1$!

Discrete-time implementation of the proposed algorithm

First-order Euler discretization with stepsize δ:

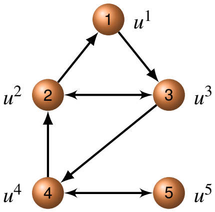
$$z^i(k+1) = z^i(k) - \delta\epsilon^{-1}\sum_{j=1}^{N} a_{ij}(z^i(k) - z^j(k))$$
$$x^i(k+1) = x^i(k) - \delta(\beta^i(x^i(k) - z^i(k)))$$

Lemma

- *Let $\mathcal{G}$ be strongly connected and weight-balanced digraph topology*
- $x^i(0) \in \mathbb{R}$ *and* $z^i(0) = u^i \in \mathbb{R}$, $\; i \in \{1, \ldots, N\}$
- *For a given* $\epsilon > 0$ *and* $\beta^i > 0$, $i \in \{1, \ldots, N\}$, *choose* $\delta \in (0, \min\{\epsilon {d^{out}_{\max}}^{-1}, \bar{\beta}^{-1}\})$, $\bar{\beta} = \max\{\beta^1, \cdots, \beta^N\}$

$$x^i(k), z^i(k) \to \frac{1}{N}\sum_{j=1}^{N} u^j \text{ as } k \to \infty, \quad \in \{1, \ldots, N\}$$

$d^{out}_{\max} = \max_{i\in\{1,\ldots,N\}}\{\sum_{i=1}^{N} a_{ij}\}$



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2. $\boldsymbol{x^i \to \frac{1}{N}\sum_{j=1}^{N} u^j, \ \ t \to \infty,}$ **even though** $\boldsymbol{\dot{x}^i = -\mathrm{sat}_{\bar{c}^i}(c^i)}$
   - **Average consensus is achieved despite limited control authority**

Think of physical processes: limited driving command

- Slow rate helps but it is not enough

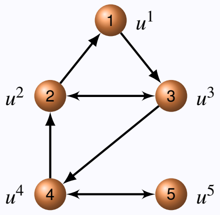
Problem Definition

$$\dot{x}^i = -c^i, \quad |\mathbf{c}^\mathbf{i}| \leqslant \bar{\mathbf{c}}^i$$

- x^i: Agreement state
- c^i: Driving command

Design $c^i = f(i, \text{neighbors})$ s.t.

$$x^i \rightarrow \frac{1}{N}\sum_{j=1}^{N} u^j, \quad t \rightarrow \infty$$



The proposed static average consensus is robust to saturation

$$\begin{cases} \epsilon\, \dot{z} = -Lz, \quad z^i(0) = u^i, \\ \dot{x}^i = -\operatorname{sat}_{\bar{c}^i}(\beta^i(x^i - z^i)), \quad x^i(0) \in \mathbb{R}, \end{cases} \qquad i\{1, \ldots, N\}$$

Lemma

$\forall\ \epsilon, \beta^i > 0, x^i(t), z^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j$, *as* $t \to \infty$.

Sketch of the proof

- $p^i = \beta(x^i - \frac{1}{N}\sum_{j=1}^N u^j), \quad q^i = -\beta^i(z^i - \frac{1}{N}\sum_{j=1}^N u^j)$
- $q^i(t)$ is a bounded and $q^i(t) \to 0$ as $t \to \infty$
- $\dot{p}^i = -\beta^i \operatorname{sat}_{\bar{c}^i}(p^i + q^i)$ is an ISS stable system (Sontag 94), i.e.,

$$p^i \to 0 \left(x^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j\right) \quad \text{as } t \to \infty$$

E. D. Sontag. On the input-to-state stability property. European Journal of Control, 1995.

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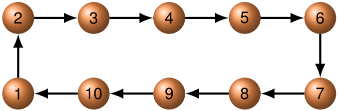
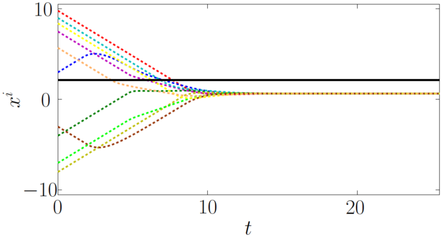
Driving command is bounded

$$\dot{x}^i = -\operatorname{sat}_{\bar{c}^i}(c^i)$$



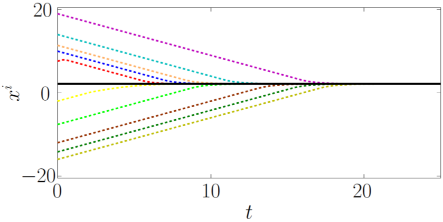
- Laplacian consensus

$$c^i = \sum_{i=1}^{N} a_{ij}(x^i - x^j)$$
$$x^i(0) = u^i,$$



- The proposed consensus

$$\begin{cases} \dot{z}^i = -\sum_{i=1}^{N} a_{ij}(x^i - x^j), \; z^i(0) = u^i, \\ c^i = x^i - z^i, \quad x^i(0) \in \mathbb{R}, \end{cases}$$



Conclusion

Summary

- We presented a distributed static average consensus algorithm which allows each agent to choose its own rate of convergence
- Our algorithm can be used to schedule the time of arrival of the agents to the agreement value
- Using our algorithm one can impose a consistent transient response over different communication topologies
- Our algorithm has intrinsic robustness against bounded driving commands
- Our algorithm is suitable for networked systems of physical processes where limited control authority exists most of the time

Future work

- Stepsize characterization for discrete-time implementation when driving command is bounded
- Extension of the results to dynamic signals.

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