Multi-stage Anti-Windup Compensation for Open-loop Stable Plants

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Abstract

e discuss the benefits of adding a measure of scheduling to the popular Anti-Windup design. The main idea is to develop a scheme in which the Anti-Windup gains depend on how much the actuator command exceeds the saturation bound. The aim is to design and implement more aggressive Anti-windup gains in lower levels of saturation. Global stability and performance is addressed by adding an outer-loop Anti-windup compensation which become active when the system is in higher levels of saturation. We present results for both static and dynamic Anti-Windup gains, along with the convex synthesis LMIs. Benefits of the proposed design method over the traditional single gain Anti-Windup compensation are demonstrated using well-known examples.

Index Terms

Saturation, Anti-Windup (AW), Scheduling, $\mathcal{L}_2$ Gain.

I. Introduction

Anti-Windup (AW) augmentation often is used for safety and performance degradations associated with actuator windup, when high performance linear controllers encounter actuator saturation. Generally, AW schemes are designed with two goals: 1) as long as system actuators do not saturate, the system closed-loop response coincides with the linear unconstrained response; 2) if the actuators saturate, stability is preserved and performance is recovered as much as possible ([1]). An excellent set of discussions and references on AW design can be found in [2], [3], [4],

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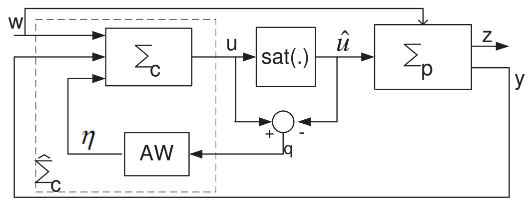

Fig. 1. Typical Anti-Windup set up

[5], [6], [7], [8], [9]. In most cases, the AW augmentation is a single controller (or set of gains) that is applied for all initial conditions, reference signals and disturbance levels. For such global results, typical performance guarantees are no better than those from the open-loop system. Naturally, stronger results can be obtained if reference signals or disturbances are assumed to have peak or energy bounds ([10]). Such techniques can be combined with the results presented here, by adding conditions that bound the reachable sets, which in turn bound the signals and states. For brevity, we focus on the global case here.

The main observation, here, is that in many applications saturation can be mostly mild and thus the command to the actuator rarely would exceed the saturation bound by a large margin. In such cases, it seems intuitively clear that using different gains for the rare cases might allow a more aggressive and higher performance gain for the situation where the commands are only slightly larger than the actuator limitations. This leads to, in essence, a form of scheduling of the AW gains based on the level of saturation. This is the main tack in this paper.

The idea of using scheduling in the traditional AW setting has been attempted before. For example, in [11] scheduling is used to improve the system performance after it re-enters the small signal domain. Another approach can be found in [12], in which a family of controllers are used to develop a scheduling approach in dealing with saturation, though the overall technique is quite different from the concept of Anti-Windup used here. While the approach in [12] has the important advantage of being applicable to open-loop unstable systems, its implementation is rather involved and requires considerable on-line computation and often cannot match the high performance of nominal linear controllers in small signal regions. Other approaches to the AW design can be found, e.g., in [13] which considers simultaneous design of linear and AW controller together, or [14] where a nonlinear state-feedback control law for the global asymptotic stabilization of non-exponentially unstable plants is proposed. Similarly, use of self

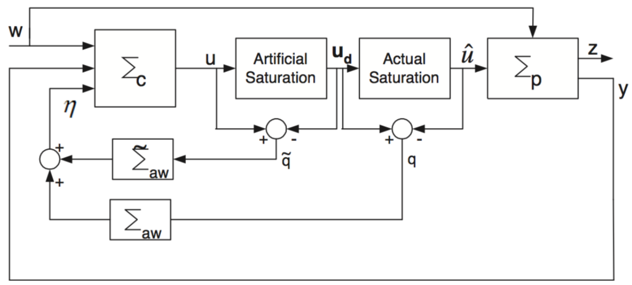

Fig. 2. Anti-Windup scheme: $u_d \in [\frac{-u_{lim}}{g_d}, \frac{u_{lim}}{g_d}]$, $\hat{u} \in [-u_{lim}, u_{lim}]$

(or gain)-scheduled techniques to obtain AW gains is explored in [15] and [16].

Here, we are mainly interested in improving the performance of AW schemes through simple and easy to implement modifications. In a somewhat related approach, in [17] a modified AW scheme is proposed which allows the saturated system to take advantage of the robustness of the nominal controller in the moderate saturation regime, and applies the AW when system faces substantial performance degradation. The main idea behind [17] was to use the high performance nominal controller even after initial saturation since, in all likelihood, a good nominal controller possesses reasonable performance robustness to the nonlinearity caused by saturation (a form of benign, 'matched' uncertainty). While this can be shown to improve the performance, it still relies on one set of AW gains for all levels of saturation.

We focus on developing an approach that provides performance guarantees for the large signal operation of open-loop stable systems, yet allows higher performance gains when the command to the actuator is only modestly above the saturation limits. This is through use of multiple sets of AW gains (as shown in Fig. 2): for moderate levels and for severe levels of saturation. By placing more weight on the gains associated with the moderate level of saturation, one can expect better performance in this level, particularly if this is the normal envelop of operation. The overall stability and some performance when the command signal exceeds the saturation bound significantly are guaranteed by another set of gains.

For simplicity, and to stay close to the basic concept of AW, we start by considering the case in which all gains are static. We show that the gains can be obtained from relatively straightforward linear matrix inequalities where the complexity of the resulting search is only modestly higher than the traditional AW case. The implementation of the proposed scheme is quite straightforward and similar to the standard AW. Finally, for 'all static' gains we show that the existence condition of the proposed scheduled approach is the same as the one for the traditional static AW compensation (Fig. 1).

It is well known however, that in some problems the traditional static AW compensation (Fig. 1) is not feasible, while a dynamic AW augmentation only requires open-loop stability, as a sufficient condition for feasibility. We show that by letting one of the loops to include dynamic AW, we also can extend the scheduling approach so that open-loop stability leads to feasibility of the scheduled AW.

II. Problem Definition and Basic Setup

Consider the following open-loop stable system with state vector $x_p \in R^{n_p}$, control input vector $\hat{u} \in R^{n_u}$, and exogenous input vector $w \in R^{n_w}$ (e.g., reference signal, noise, external disturbance, etc.):

$$\Sigma_p \quad \sim \quad \begin{cases} \dot{x}_p = A_p x_p + B_1 w + B_2 \hat{u} \\ y = C_2 x_p + D_{21} w + D_{22} \hat{u} \\ z = C_1 x_p + D_{11} w + D_{12} \hat{u} \end{cases} . \tag{1}$$

The *nominal* controller, designed to fulfill a specific task such as tracking or disturbance regulation for this system, is likely to saturate. As a result, it needs to be augmented with an AW protection loop. The AW commands are introduced to the nominal controller by adding signals η_1 and η_2 to the state and output equations of the nominal controller (i.e., the nominal controller is recovered when $\eta_1 = \eta_2 = 0$):

$$\Sigma_c \quad \sim \quad \begin{cases} \dot{x}_c = A_c x_c + B_{cy} y + B_{cw} w + \eta_1 \\ u = C_c x_c + D_{cy} y + D_{cw} w + \eta_2 \end{cases} \tag{2}$$

where $x_c(t) \in R^{n_c}$ is the controller state vector and $u \in R^{n_u}$ is the output vector of the controller. The saturation nonlinearity is assumed decentralized with the saturation limit u_{lim} for each u_i $(i = 1, 2, ..., n_u)$; i.e., $\hat{u}_i = sat(u_i) = sgn(u_i)\, min\{|u_i|, u_{lim}\}$.

In traditional AW scheme (Fig. 1), AW commands are normally generated using only one set of gains designed to ensure global stability and graceful performance degradation even for arbitrary large commands/disturbances. In this paper, we use more than one set of gains. As a start, we consider the case where two sets of gains are obtained: one for the case when the control command is moderately above the saturation limit, and another when the control command is significantly larger.

Consider Figure 2. To separate the activation of AW gains into two different levels of saturation, we have added an artificial saturation box with a larger saturation bound, $\frac{1}{g_d}u_{lim}$ where $0 < g_d \leq 1$. For moderate levels of saturation, i.e., when the magnitude of the command signal is between u_{lim} and $\frac{1}{g_d}u_{lim}$, we use AW gain Σ_{aw} and the signal to activate this AW gain is $q = u_d - \hat{u}$, where u_d is the output of the artificial saturation box. If the actuator command goes beyond $\frac{1}{g_d}u_{lim}$, then $\tilde{q} = u - u_d \neq 0$ and both q and $\tilde{q}$ activate their respective AW gains, Σ_{aw} and $\tilde{\Sigma}_{aw}$.

The main benefit of the first, artificial, saturation box is to bound the sector nonlinearity, associate with the actual saturation box, away from 1 since it limits the magnitude of the signal which will enter the actual saturation u_d (associated with the actuator) to $\frac{1}{g_d}u_{lim}$. Then, it is straightforward to show that

$$(q, u_d) \in [0, s_d] \Longrightarrow q^T(q - s_d u_d) \leq 0, \text{ where } \; s_d = 1 - g_d \tag{3}$$

Absent any assumptions on the reference and disturbances signals, there is no a priori bound on the signal u. Therefore, the sector bounds on the artificial saturation is

$$(\tilde{q}, u) \in \; [0, 1] \Longrightarrow \tilde{q}^T(\tilde{q} - u) \leq 0 \tag{4}$$

The two sector conditions separate the task of dealing with moderate signals to Σ_{aw} where it deals with a nonlinear element with more limited sector nonlinearity, while substantially larger signals will also activate $\tilde{\Sigma}_{aw}$. One can expect that this will allow a more effective and aggressive Σ_{aw} for moderately saturated command signals (see Table 1 of [18] and the numerical examples below). Now considering the signals in Fig. 2, we have $q = u_d - \hat{u}$ and $\tilde{q} = u - u_d$, thus $u_d = u - \tilde{q}$, and $\hat{u} = u - (q + \tilde{q})$, while the AW signals entering the compensator dynamics; i.e., η_1 and η_2, satisfy $\eta = +\tilde{\Sigma}_{aw}(\tilde{q}) + \Sigma_{aw}(q)$, with $\eta = [\eta_1^T \;\; \eta_2^T]^T$. Then, the plant and controller equations can be written as:

$$\Sigma_p \quad \sim \quad \begin{cases} \dot{x}_p = A_p x_p + B_1 w + B_2(u - (q + \tilde{q})) \\ y = C_2 x_p + D_{21} w + D_{22}(u - (q + \tilde{q})) \\ z = C_1 x_p + D_{11} w + D_{12}(u - (q + \tilde{q})) \end{cases} \tag{5}$$

$$\Sigma_c \sim \begin{cases} \dot{x}_c = A_c x_c + B_{cy} y + B_{cw} w + [I \quad 0](\tilde{\Sigma}_{aw}(\tilde{q}) + \Sigma_{aw}(q)) \\ u = C_c x_c + D_{cy} y + D_{cw} w + [0 \quad I](\tilde{\Sigma}_{aw}(\tilde{q}) + \Sigma_{aw}(q)) \end{cases} \tag{6}$$

In the next section, we focus on the static compensation. Dynamic gains are discussed in Section IV. Also, in the following, for simplicity, we mostly consider the single actuator systems first. The expansion to the multi-actuator case is straightforward and will be discussed later.

III. Static AW gains

A. Synthesis

The simplest AW form in terms of design and implementation is static gain. Here, we obtain synthesis LMIs for the scheduled AW scheme of Fig. 2 when Σ_{aw} and $\tilde{\Sigma}_{aw}$ are considered static gains $-\Lambda$ and $-\tilde{\Lambda}$, respectively (i.e. $\Sigma_{aw}(q) = -\Lambda q$ and $\tilde{\Sigma}_{aw}(\tilde{q}) = -\tilde{\Lambda}\tilde{q}$). In this case, by selecting $x = [x_p^T \quad x_c^T]^T$ and using w, q and $\tilde{q}$ as input, we can represent the augmented closed-loop system, Σ, in the following state space form:

$$\begin{cases} \dot{x} = & Ax + B_w w + (B_q - B_\eta \Lambda) q + (B_q - B_\eta \tilde{\Lambda})\tilde{q} \\ z = & C_z x + D_{zw} w + (D_{zq} - D_{z\eta}\Lambda) q + (D_{zq} - D_{z\eta}\tilde{\Lambda})\tilde{q} \\ u = & C_u x + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda) q + (D_{uq} - D_{u\eta}\tilde{\Lambda})\tilde{q} \end{cases} \tag{7}$$

The matrices A, B_w, etc. are known matrices in terms of plant and nominal controller gains (see Appendix I). To establish a performance bound for the AW and ensure stability, $\mathcal{L}_2$ gain from w to z is typically considered. A variety of other measures, such as energy-to-peak or peak-to-peak, can be handled easily but for brevity we discuss only the $\mathcal{L}_2$ gain. Following the standard approach, by relying on quadratic Lyapunov function $V = x^T Q^{-1} x$ with $Q > 0$, an estimate for the $\mathcal{L}_2$ gain from w to z, $\tilde{\gamma}$, can be obtained using the following standard inequality:

$$\frac{d}{dt}(x^T Q^{-1} x) + \tilde{\gamma}^{-1} z^T z - \tilde{\gamma} w^T w < 0 \tag{8}$$

Here, the two nonlinear elements satisfy the sector conditions in (3) and (4). Invoking S-procedure for some positive scalar $\tilde{W}$ and W_o, we can use the following sufficient condition to ensure (8):

$$\begin{aligned} \tfrac{d}{dt}(x^T Q^{-1} x) + \tilde{\gamma}^{-1} z^T z - \tilde{\gamma} w^T w - 2q^T W_o (q - s_d u_d) \\ -2\tilde{q}^T \tilde{W}(\tilde{q} - u) < 0 \end{aligned} \tag{9}$$

Since $u_d = u - \tilde{q}$, and using $W = W_o s_d$, we can rewrite (9)

$$\frac{d}{dt}(x^T Q^{-1} x) + \tilde{\gamma}^{-1} z^T z - \tilde{\gamma} w^T w - 2q^T W(s_d^{-1} q - u + \tilde{q}) \\ -2\tilde{q}^T \tilde{W}(\tilde{q} - u) < 0$$

Using (7), this inequality can be expanded into the typical form of $p^T(H + \tilde{\gamma}^{-1} h h^T)p < 0$ where $p^T = (x^T \ w^T \ q^T \ \tilde{q}^T)$ with a sufficient condition of $H + \tilde{\gamma}^{-1} h h^T < 0$ (details are routine and are omitted). Applying the Schur complement formula, one gets the equivalent form of $\Omega = \begin{pmatrix} H & h \\ h^T & -\tilde{\gamma} \end{pmatrix} < 0$. Next, we apply the congruent transformation of $T^T \Omega T$ with $T = Diag[I, I, I, W^{-1}, \tilde{W}^{-1}]$. After defining $\mathbf{M} = \mathbf{W}^{-1}$, $\tilde{\mathbf{M}} = \mathbf{W}^{-1}$, $\tilde{\mathbf{X}} = \tilde{\mathbf{\Lambda}}\tilde{\mathbf{M}}$ and $\mathbf{X} = \mathbf{\Lambda M}$, we get

$$\begin{pmatrix} A\mathbf{Q} + \mathbf{Q}^T A & \star & \star & \star & \star \\ B_w^T & -\tilde{\gamma} I & \star & \star & \star \\ C_z \mathbf{Q} & D_{zw} & -\tilde{\gamma} I & \star & \star \\ \Phi_{4,1} & D_{uw} & \mathbf{M} D_{zq}^T - \mathbf{X}^T D_{z\eta}^T & \Phi_{4,4} & \star \\ \Phi_{5,1} & D_{uw} & \tilde{\mathbf{M}} D_{zq}^T - \tilde{\mathbf{X}}^T D_{z\eta}^T & \Phi_{5,4} & \Phi_{5,5} \end{pmatrix} < 0 \tag{10}$$

where

$$\Phi_{4,1} = \mathbf{M} B_q^T - \mathbf{X}^T B_\eta^T + C_u \mathbf{Q}, \quad \Phi_{5,1} = \tilde{\mathbf{M}} B_q^T - \tilde{\mathbf{X}}^T B_\eta^T + C_u \mathbf{Q}$$
$$\Phi_{4,4} = He\{-s_d^{-1}\mathbf{M} + D_{uq}\mathbf{M} - D_{u\eta}\mathbf{X}\}$$
$$\Phi_{5,4} = -\tilde{\mathbf{M}} + D_{uq}\mathbf{M} + \tilde{\mathbf{M}} D_{uq}^T - D_{u\eta}\mathbf{X} - \tilde{\mathbf{X}}^T D_{u\eta}^T$$
$$\Phi_{5,5} = -2\tilde{\mathbf{M}} + D_{uq}\tilde{\mathbf{M}} + \tilde{\mathbf{M}} D_{uq}^T - D_{u\eta}\tilde{\mathbf{X}} - \tilde{\mathbf{X}}^T D_{u\eta}^T$$

The sub-block (4:5,4:5) ensures the well-posedness of the closed-loop system under the scheme of Fig. 2 (see [19] for details). By minimizing $\tilde{\gamma}$ with the constraint (10), we obtain the stabilizing gains with guaranteed performance level of $\tilde{\gamma}$ as

$$\tilde{\Lambda} = \tilde{\mathbf{X}}\tilde{\mathbf{M}}^{-1}, \quad \Lambda = \mathbf{X}\mathbf{M}^{-1} \tag{11}$$

One can put more emphasis on the performance level in the moderate levels of saturation (i.e., $|u| \leq \frac{1}{g_d} u_{lim}$) by the following approach. We start by assuming, *for now*, that somehow it can be guaranteed (without using an artificial saturation element) that the command to the actuator u_d satisfies $|u_d(t)| \leq \frac{1}{g_d} u_{lim}$ (or, equivalently, $u(t)$ in Fig. 1 satisfies this bound). In that case, we could design a more aggressive AW gain Λ, since, now, there is no $\tilde{\Lambda}$ loop. Parallel to development above using $x = [x_p^T \quad x_c^T]^T$ with w and q as inputs, we can represent the closed-loop system (1), (2), with $\eta = -\Lambda q$, in moderate levels of saturation by (7) where $\tilde{q}$ is set to zero. To establish a performance bound for the AW and ensure stability, we again rely on quadratic Lyapunov function $V = x^T Q^{-1} x$. However, as we discussed, if the magnitude bound is in effect, we have $(q, u_d) \in [0, s_d]$. As a result, we have $q^T W_o(q - s_d u) < 0$ where $W_o > 0$. Invoking S-procedure, inequality

$$\frac{d}{dt}(x^T Q^{-1} x) + \gamma^{-1} z^T z - \gamma w^T w - 2q^T W_o(q - s_d u) < 0 \tag{12}$$

is the sufficient condition for inequality (8). Expanding this inequality and preforming proper congruent transformations, inequality (12) can be written in the LMI below with $\mathbf{M} = (s_d \mathbf{W}_o)^{-1}$ and $\mathbf{X} = \mathbf{\Lambda M}$

$$\begin{pmatrix} A\mathbf{Q} + \mathbf{Q}^T A & \star & \star & \star \\ B_w^T & -\gamma I & \star & \star \\ C_z \mathbf{Q} & D_{zw} & -\gamma I & \star \\ \Phi_{4,1} & D_{uw} & \mathbf{M} D_{zq}^T - \mathbf{X}^T D_{z\eta}^T & \Phi_{4,4} \end{pmatrix} < 0\,. \tag{13}$$

Of course, using inequality (13) alone does not guarantee the stability and performance since it is based on the assumption on the magnitude of the command. One way to ensure overall stability and performance is to combine the two inequalities (13) and (10). To ensure a unique Λ, we need to use the same $\mathbf{M}$ and $\mathbf{X}$ in (13) and (10), though we can use different Lyapunov matrix in (13) to reduce conservatism (denoted by $\bar{\mathbf{Q}}$). This results in the following:

$$\text{Minimize} \quad \tilde{c}\,\tilde{\gamma} + c\,\gamma \;(\text{with} \;\; \tilde{c} > 0, c > 0) \;\; \text{subject to} \tag{14}$$

$$\text{LMI } (10) \;\; \text{and} \tag{15}$$

$$\begin{pmatrix} A\bar{\mathbf{Q}} + \bar{\mathbf{Q}}^T A & \star & \star & \star \\ B_w^T & -\gamma I & \star & \star \\ C_z \bar{\mathbf{Q}} & D_{zw} & -\gamma I & \star \\ \mathbf{M} B_q^T - \mathbf{X}^T B_\eta^T + C_u \bar{\mathbf{Q}} & D_{uw} & \mathbf{M} D_{zq}^T - \mathbf{X}^T D_{z\eta}^T & \Phi_{4,4} \end{pmatrix} < 0 \tag{16}$$

Inequality (15), as before, ensures that the closed-loop in Fig. (2) is stable with an $\mathcal{L}_2$ gain of $\tilde{\gamma}$, while (16) indicates that the $\mathcal{L}_2$ gain would have been γ if the command magnitude remains below $\frac{1}{g_d}u_{lim}$. Buy using larger values for c, one can seek lower values for γ (i.e., more aggressive AW for moderate saturation cases), at a cost of larger $\tilde{\gamma}$ which is the guaranteed $\mathcal{L}_2$ gain of the closed-loop. It is important to note that the only performance guarantee, i.e., the one without any assumptions or conditions, is $\tilde{\gamma}$. The gain γ is best described as a measure of the aggressiveness and effectiveness of Λ.

Remark 1. *The levels of saturation are defined and separated by the choice of s_d (or g_d). Smaller values of s_d result in less restrictive sector condition for the inner loop and therefore, one can expect to obtain more aggressive gains for inner loop. This design parameter can be selected based on the operating envelop of the system, e.g., if the saturated system is more often in the lower levels then we can pick a small value for s_d and get aggressive gains (higher performance) in this region.*

Remark 2. *For multi-input systems, one only needs to replace the sector end for the moderate levels of saturation, s_d, in all of the equations by the diagonal matrix S_D, where each diagonal entry is $s_{d_i} = 1 - g_{d_i}$ with g_{d_i} the design point for artificial saturation elements, while the weights W and $\tilde{W}$ will be diagonal positive definite matrices. Furthermore, it is relatively straightforward to see that one can insert additional artificial boxes, which will result in additional rows and columns in the main inequality in (10). Additional saturation elements can help refine the range of the over-saturations that each AW gain will handle, and place emphases on a specific range of operation by following the development in (14)-(16), in which (16) is replaced by inequality (or inequalities) that correspond to the range (or ranges) of interest.*

B. Feasibility

Next, we consider the conditions under which the proposed scheme is guaranteed to have solutions. First, note that with $\bar{Q} = Q$ and $\gamma = \tilde{\gamma}$ the inequality (16) is simply (1:4,1:4) bock of (10). As a result, if (10) is feasible, then there is at least one set of decision variables that make (14)-(16) feasible. The arrangement in (14)-(16) is simply to allow a more aggressive Λ, in exchange for a higher guaranteed $\mathcal{L}_2$ gain (see examples below). To consider the feasibility of (10), similar to the traditional AW, we use the elimination lemma. It is relatively easy to see that (10) can be written as $\Psi + G^T[\tilde{X} \;\; X]^T H + H^T[\tilde{X} \;\; X]G < 0$, where $H = [-B_\eta^T \; 0 \; -D_{z\eta}^T \; -D_{u\eta}^T \; -D_{u\eta}^T \,]$ and $G = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$. Applying the standard elimination lemma, we get the following equivalent conditions

$$\begin{pmatrix} A\mathbf{Q} + \mathbf{Q}A^T & B_w & \mathbf{Q}C_z^T \\ B_w^T & -\tilde{\gamma}I & D_{zw}^T \\ C_z\mathbf{Q} & D_{zw} & -\tilde{\gamma}I \end{pmatrix} < 0 \tag{17}$$

$$\begin{pmatrix} \Upsilon_{1,1} & \star & \star \\ B_1^T & -\tilde{\gamma}I & \star \\ C_1\mathbf{Q}_{11} & D_{11} & -\tilde{\gamma}I \end{pmatrix} + \frac{s_d}{2}\begin{pmatrix} B_2 \\ 0 \\ D_{12} \end{pmatrix} \mathbf{M} \begin{pmatrix} B_2^T & 0 & D_{12}^T \end{pmatrix} < 0 \tag{18}$$

where $\Upsilon_{1,1} = A_p\mathbf{Q}_{11} + \mathbf{Q}_{11}A_p^T$. Consider (18) first. If the first term is negative definite, then there exists small enough $\mathbf{M}$ such that (18) holds. If the open-loop system is stable, then a $\mathbf{Q}_{11} > 0$ (top right hand block of $\mathbf{Q}$) exists so that the first terms holds for some $\tilde{\gamma}$. Similarly, stability of closed-loop means that there exists some $\mathbf{Q} > 0$ such that (17) holds. For both (17) and (18) to hold together, $\mathbf{Q}_{11}$ must be the (1,1) block of $\mathbf{Q}$. These are precisely the sufficient and necessary conditions for the existence of a single static AW gain (see [6] for more details). That is, if the traditional AW problem has a solution, the scheduled scheme proposed here (in (10) and indeed for (14)-(16)) is also feasible. As with the traditional case, the choice of $\tilde{M}$ does not play a role, while M is to be small enough to make (18) feasible.

Of course, it is well known that this ‘common Q_{11} block’ requirement is not possible in many problems. Dynamic AW gains address this difficulty in the traditional AW approach. Next, we show that by using dynamic AW for the first (artificial saturation) loop, we can address this issue for the scheduling scheme proposed here.

IV. Combination of Dynamic and Static AW

Our focus in this work is to keep the multiple gain AW scheme as simple as possible both in design and implementation stages. As mentioned above, the motivation to move from static gains to more complex dynamic forms is the infeasibility problem associated with ‘all static’ gains for some systems. In the following, we show that by letting only one of the loops to include dynamic AW (the outer loop), one can also extend the scheduling approach so that open-loop stability alone leads to feasibility of the scheduled AW. Of course, one can replace both static gains with dynamic ones, but this will add to the complexity of the design and implementation, without additional benefits.

A. Synthesis

Consider the case where the first, artificial, AW loop has dynamic gains while the second one has only static gains, i.e., we use the following structure for Σ_{aw} and $\widetilde{\Sigma}_{aw}$ in Fig. 2

$$\Sigma_{aw} \sim \; -\Lambda q \;\; , \quad \widetilde{\Sigma}_{aw} \sim \begin{cases} \dot{x}_a = A_a x_a + B_a \tilde{q} \\ \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix} = C_a x_{aw} + D_a \tilde{q} \end{cases} \tag{19}$$

where $x_a \in R^{n_a}$. In the following, we will use the notation $\tilde{\Lambda} = -\begin{bmatrix} B_a \\ D_a \end{bmatrix}$. Defining $x = [x_p^T \;\; x_c^T]^T$ and $\tilde{x} = [x^T \;\; x_a^T]^T$ with w, q and $\tilde{q}$ as inputs, we can represent the augmented closed-loop system, Σ, as follows:

$$\begin{cases} \dot{\tilde{x}} = & \tilde{\mathbf{A}}\tilde{x} + \tilde{B}_w w + (\tilde{B}_q - \tilde{B}_\eta \Lambda) q + (\tilde{B}_q - \tilde{B}_{\tilde{\eta}} \tilde{\Lambda}) \tilde{q} \\ z = & \tilde{\mathbf{C}}_z \tilde{x} + D_{zw} w + (D_{zq} - D_{z\eta}\Lambda) q + (D_{zq} - D_{z\tilde{\eta}} \tilde{\Lambda}) \tilde{q} \\ u = & \tilde{\mathbf{C}}_u \tilde{x} + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda) q + (D_{uq} - D_{u\tilde{\eta}} \tilde{\Lambda}) \tilde{q} \end{cases} . \tag{20}$$

Matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{C}}_u$ and $\tilde{\mathbf{C}}_z$ contain the dynamic AW matrices A_a and/or C_a. By relying on $V = \tilde{x}^T Q^{-1} \tilde{x}$ with $Q > 0$, and following the same steps in pervious section, the $\mathcal{L}_2$ gain from w to z (i.e., $\tilde{\gamma}$) can be obtained from matrix inequality (10) when the system matrices A, B_w, etc are replaced with their tilde counterparts in (20) – i.e., $\tilde{A}$, etc. Once again, the sub-block (4:5,4:5) ensures the well-posedness of the closed-loop system under the scheduled AW scheme of Fig. 2 (see [20]). By minimizing $\tilde{\gamma}$ with respect to (10), we get the stabilizing Σ_{aw} and $\widetilde{\Sigma}_{aw}$ with guaranteed performance level of $\tilde{\gamma}$.

Since the matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{C}}_z$, and $\tilde{\mathbf{C}}_u$ contain the AW matrices A_a and/or C_a, the inequality (10) in its current representation does not form a convex problem. However, following an argument similar to those in [6], we can show that one can convexify the synthesis matrix inequalities above for any dynamic AW of order $n_a \geq n_p$ (see Subsection IV-B). Ref. [6] also shows that any performance level obtained by higher orders can be obtained by $n_a = n_p$. However, in [6], the solution of $n_a = n_p$ is not explicit and involves a four-step procedure. In order to develop an approach that can be used in other extensions (such as scheduling, or multi-objective problems), we use the change of variable approach in [21]. While this yields a relatively simple approach, it leads to a dynamic AW with $n_a = n_p + n_c$. For the change of the variable approach, without any loss of generality, we use the following Lyapunov matrix:

$$Q = \begin{bmatrix} Y & S \\ S & S \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} Z & -Z \\ -Z & Z + S^{-1} \end{bmatrix} \tag{21}$$

with $Y = Y^T,\ S = S^T,\ Z = Z^T \in R^{(n_p+n_c)\times(n_p+n_c)}$. We define the new variables $F_1 = A_a S$, $F_2 = C_a S$, $F_3 = B_a \tilde{M}$, $F_4 = D_a \tilde{M}$ and $X = \Lambda M$. Then, we can obtain the stabilizing Σ_{aw} and $\widetilde{\Sigma}_{aw}$ with guaranteed performance level of $\tilde{\gamma}$ from the theorem below.

Theorem 1. *The closed-loop system in (20) is stable and the $\mathcal{L}_2$ gain from w to z is less than $\tilde{\gamma}$, if there exist positive scalars M, $\tilde{M}$, symmetric matrices $\mathbf{Y} > 0$ and $\mathbf{S} > 0$, and matrices $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ satisfying*

$$\begin{pmatrix} \Omega_{1,1} & \star & \star & \star & \star & \star \\ \Omega_{2,1} & \mathbf{F}_1 + \mathbf{F}_1^T & \star & \star & \star & \star \\ B_w^T & 0 & -\tilde{\gamma} I & \star & \star & \star \\ \Omega_{4,1} & C_z\mathbf{S} + D_{z\eta}\mathbf{F}_2 & D_{zw} & -\tilde{\gamma} I & \star & \star \\ \Omega_{5,1} & C_u\mathbf{S} + D_{u\eta}\mathbf{F}_2 & D_{uw} & \Omega_{54} & \Omega_{55} & \star \\ \Omega_{6,1} & \mathbf{F}_3^T + C_u\mathbf{S} + D_{u\eta}\mathbf{F}_2 & D_{uw} & \Omega_{64} & \Omega_{65} & \Omega_{66} \end{pmatrix} < 0 \tag{22}$$

where

$$\Omega_{11} = A\mathbf{Y} + \mathbf{Y}A^T + B_\eta \mathbf{F}_2 + \mathbf{F}_2^T B_\eta^T, \quad \Omega_{41} = C_z\mathbf{Y} + D_{z\eta}\mathbf{F}_2$$
$$\Omega_{21} = \mathbf{S}A^T + \mathbf{F}_2^T B_\eta^T + \mathbf{F}_1,$$
$$\Omega_{51} = \mathbf{M}B_q^T - \mathbf{X}^T B_\eta^T + C_u\mathbf{Y} + D_{u\eta}\mathbf{F}_2,$$
$$\Omega_{54} = \mathbf{M}D_{zq}^T - \mathbf{X}^T D_{z\eta}^T, \quad \Omega_{64} = \tilde{\mathbf{M}}D_{zq}^T - \tilde{\mathbf{X}}^T D_{z\tilde{\eta}}^T,$$
$$\Omega_{55} = -2s_d^{-1}\mathbf{M} + D_{uq}\mathbf{M} + \mathbf{M}D_{uq}^T - D_{u\eta}\mathbf{X} - \mathbf{X}^T D_{u\eta}^T,$$
$$\Omega_{61} = \tilde{\mathbf{M}}B_q^T + \mathbf{F}_4^T B_\eta^T + C_u\mathbf{Y} + D_{u\eta}\mathbf{F}_2,$$
$$\Omega_{65} = -\tilde{\mathbf{M}} + D_{uq}\mathbf{M} + \tilde{\mathbf{M}}D_{uq}^T - D_{u\eta}\mathbf{X} + \mathbf{F}_4^T D_{u\eta}^T,$$
$$\Omega_{66} = -2\tilde{\mathbf{M}} + D_{uq}\tilde{\mathbf{M}} + \tilde{\mathbf{M}}D_{uq}^T + D_{u\eta}\mathbf{F}_4 + \mathbf{F}_4^T D_{u\eta}^T.$$

The system matrices A, B_η, etc. are defined in Appendix I. Once this optimization problem is solved, the AW gains are $A_a = \mathbf{F}_1\mathbf{S}^{-1}$, $B_a = \mathbf{F}_3\tilde{\mathbf{M}}^{-1}$, $C_a = \mathbf{F}_2\mathbf{S}^{-1}$, $D_a = \mathbf{F}_4\tilde{\mathbf{M}}^{-1}$, and

$\Lambda = \mathbf{X}\mathbf{M}^{-1}$.

To put more emphasis on the performance level in the moderate levels of saturation (i.e., $|u| \leq \frac{1}{g_d} u_{lim}$), we use an approach similar to the 'all static' case. As before, we start by assuming somehow it can be guaranteed that the command to the actuator u_d satisfies $|u_d(t)| \leq \frac{1}{g_d} u_{lim}$. In that case, we could design a more aggressive AW gain Λ. For this, we consider the case that the first saturation element is not saturated; i.e., when $\tilde{\Sigma}$ exists but $\tilde{q} = 0$. In this case, we can assume B_a and D_a are zero, but A_a and C_a have to be accounted for (think initial conditions and transients). As a result, A and C_z matrices will contain the dynamic AW matrices A_a and C_a.

Parallel to earlier development, by selecting $\tilde{x} = [x^T \;\; x_a^T]^T$ and considering w and q as inputs, we can represent the closed-loop system in moderate levels of saturation by (20) while $\tilde{q}$ is set to zero. Note that in this case we can now set D_a and B_a to zero to simplify the resulting inequalities (while keeping the same feasibility conditions, see below). To establish a performance bound for the AW and ensure stability, we again rely on quadratic Lyapunov function $V = \tilde{x}^T Q^{-1} \tilde{x}$. As before, if the magnitude bound is in effect, we have $(q, u_d) \in [0, s_d]$, and $q^T W (q - s_d u) < 0$ where $W > 0$ is a scale. Then invoking S-procedure for some $W > 0$, it is easy to see that (12) is the sufficient condition for inequality (8). By expanding (12) in terms of system matrices and after proper congruent transformations, we obtain:

$$\begin{pmatrix} \Omega_{1,1} & \star & \star & \star & \star \\ \Omega_{2,1} & \mathbf{F}_1 + \mathbf{F}_1^T & \star & \star & \star \\ B_w^T & 0 & -\gamma I & \star & \star \\ \Omega_{4,1} & C_z \mathbf{S} + D_{z\eta} \mathbf{F}_2 & D_{zw} & -\gamma I & \star \\ \Omega_{5,1} & C_u \mathbf{S} + D_{u\eta} \mathbf{F}_2 & D_{uw} & \Omega_{5,4} & \Omega_{55} \end{pmatrix} < 0 \tag{23}$$

Similar to case of static gains, inequality (23) alone does not guarantee the stability and performance since it is based on the assumption on the magnitude of the command. One way to ensure overall stability and performance is to combine (23) and (22). In both approaches, since the same set of AW gains are implemented, certain variables have to be common, i.e. the two inequalities (22) and (23) must have the same $\mathbf{M}$, $\mathbf{X}$, $\mathbf{S}$, $\mathbf{F}_1$ and $\mathbf{F}_2$, but in (23), we can use a different $\mathbf{Y}$ ($\bar{\mathbf{Y}}$):

$$\text{Minimize } \tilde{c}\,\tilde{\gamma} + c\,\gamma \text{ with } \ (\tilde{c} > 0, \ \ c > 0) \text{ subject to} \tag{24}$$

$$\text{LMI } (22) \ \text{ and} \tag{25}$$

$$\begin{pmatrix} \bar{\Omega}_{1,1} & \star & \star & \star & \star \\ \Omega_{21} & \mathbf{F}_1 + \mathbf{F}_1^T & \star & \star & \star \\ B_w^T & 0 & -\gamma I & \star & \star \\ \bar{\Omega}_{4,1} & C_z\mathbf{S} + D_{z\eta}\mathbf{F}_2 & D_{zw} & -\gamma I & \star \\ \bar{\Omega}_{5,1} & C_u\mathbf{S} + D_{u\eta}\mathbf{F}_2 & D_{uw} & \Omega_{54} & \Omega_{55} \end{pmatrix} < 0 \tag{26}$$

where $\bar{\Omega}$'s are the counterpart Ω's when Y is replaced with $\bar{Y}$.

Inequality (25), as before, ensures that the closed loop in Fig. (2) is stable with an $\mathcal{L}_2$ gain of $\tilde{\gamma}$, while (26) indicates that the $\mathcal{L}_2$ gain would have been γ if the command magnitude remains below $\frac{1}{g_d}u_{lim}$. As with the case of all static gains one can use $\tilde{c}$ and c to obtain a more aggressive γ. Recall that the only performance guarantee is $\tilde{\gamma}$, while γ is best described as a measure of the aggressiveness and effectiveness of Λ (the lower it is, the more effective Λ should be in moderate levels of saturation).

B. *Feasibility Condition for Dynamic-Static Scheduled Scheme*

First, note that as in the case of static compensation, with common variables, (26) is the (1:5,1:5) block of (22), so the search in (24-26) will always have at least one solution if (22) holds. We thus need only to focus on (22). Consider inequality (10) when the outer loop is dynamic AW, i.e, when the system matrices A, B_w, etc are replaced with their tilde counterparts shown in (20). In the resulting matrix inequality, let us substitute for the Lyapunov matrix Q the following partitioned form:

$$Q = \begin{bmatrix} R & V \\ V^T & Z \end{bmatrix}, \quad Q^{-1} = P = \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

where $R, S \in R^{(n_p+n_c)\times(n_p+n_c)}, \ Z, P_{22} \in R^{n_a\times n_a}, \ V, P_{12} \in R^{(n_p+n_c)\times n_a}$. It is relatively easy to see that such an inequality can be expressed as $\Psi + He\{G^T \begin{bmatrix} A_a & F_3 & \odot \\ C_a & F_4 & -X \end{bmatrix}^T H\} < 0$, where

$\odot$ could be any arbitrary matrix of appropriate dimension where

$$H = \begin{bmatrix} 0 & I_{n_a} & 0 & 0 & 0 & 0 \\ B_\eta^T & 0 & 0 & D_{u\eta}^T D_{12}^T & D_{u\eta}^T & D_{u\eta}^T \end{bmatrix}, \text{ and}$$

$$G = \begin{bmatrix} V^T & Z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_u} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_u} \end{bmatrix} =$$

$$\left[\begin{array}{cc|c|c|c|c} 0 & I_{n_a} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n_u} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{n_u} \end{array}\right] Diag\left[\begin{array}{ccccc} Q, & I, & I, & I, & I \end{array}\right].$$

Applying the elimination lemma, we get the following equivalent conditions for $n_a \geq n_p$

$$\begin{pmatrix} \Gamma_{1,1} & \star & \star \\ B_1^T & -\gamma I & \star \\ C_1 \mathbf{R}_{11} & D_{11} & -\gamma I \end{pmatrix} + \frac{s_d}{2} \begin{pmatrix} B_2 \\ 0 \\ D_{12} \end{pmatrix} \mathbf{M} \begin{pmatrix} B_2^T & 0 & D_{12}^T \end{pmatrix} < 0 \tag{27}$$

$$\begin{pmatrix} A\mathbf{S} + \mathbf{S}A^T & B_w & \mathbf{S}C_z^T \\ B_w^T & -\gamma I & D_{uw}^T \\ C_z \mathbf{S} & D_{uw} & -\gamma I \end{pmatrix} < 0 \tag{28}$$

$$\mathbf{S} = \mathbf{S}^T = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^T & \mathbf{S}_{22} \end{bmatrix} > 0, \quad \mathbf{R}_{11} = \mathbf{R}_{11}^T > 0, \quad \mathbf{R}_{11} - \mathbf{S}_{11} > 0 \tag{29}$$

where $\Gamma_{1,1} = A_p \mathbf{R}_{11} + \mathbf{R}_{11} A_p^T$.

Details, including the non-convex conditions for $n_a < n_p$, follow [6] closely and are not repeated. The key is that the feasibility conditions become stability of the open loop and nominal closed loop, the same as the ones associated with a single dynamic AW gains (as in Fig. 1). The only difference is the second term in (27) which can be made arbitrarily small with M. Adding the second saturation element thus does not weaken the existence conditions, while allowing for a more aggressive AW action.

V. Numerical Examples

A. Static only gains

Consider the following system taken from [1] with input bound $u_{lim} = 1$

$$\left[\begin{array}{c|c|c} A_p & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \end{array}\right] = \left[\begin{array}{ccc|c|c} -10.6 & -6.09 & -0.9 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & 0 \end{array}\right],$$

with $z = y - w$ (w is the reference signal in this example). and nominal controller

$$\left[\begin{array}{c|c|c} A_c & B_c y & B_{cr} \\ \hline C_c & D_{cy} & D_{cr} \end{array}\right] = \left[\begin{array}{cc|c|c} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array}\right].$$

Nominal γ in this problem is 1. Using the results of [5] or [17], the static AW gain from the traditional approach is $\Lambda = [-0.1968 \quad 0.0025 \quad -0.9860]^T$which leads to a performance level of $\gamma = 85.78$. For this system we choose $s_d = 0.2$. Using the proposed technique with $\tilde{c} = 1$ and $c = 200$, AW gains are $\Lambda = 10^3[-1.5120 \quad 0.0189 \quad 0.0795]^T$ and $\tilde{\Lambda} = [-97.3641 \quad 1.2207 \quad 0.9435]^T$. This design leads to $\tilde{\gamma} = 1062.5$ and $\gamma = 5.931$. Figure 3 shows the results of a simulation with a small ((a)) and a large ((b)) reference signal for this design. As these figures illustrate, the scheduled system shows better response than the system with single AW gain (traditional AW), especially for small signal case (the actuator is saturated in both cases). Figure 3 also shows the results of a simulation with the same small ((c)) and large ((d)) reference signals for a design with $\tilde{c} = c = 1$. In this case, we obtain $\tilde{\gamma} = 98.9$ and $\gamma = 67.92$ and gains that are in the same order or magnitude as the case with $c = 200$.

Figure 4 shows the time histories of u, u_d and $\hat{u}$ for the first two cases discussed above (i.e., when $\tilde{c} = 1$, and $c = 200$). As this set of figures shows for small reference signal, the outer-loop never gets active. Therefore, the system is applying a higher performance AW compensation compared to the traditional AW, yielding a better response. For larger signal both gains are active. Similar behavior is observed with AW gains corresponding to $\tilde{c} = c = 1$.

In both figures, the ideal (without saturation limitations) tracks the input almost perfectly, which is not surprising, and the plot of the output for this idealized case would be very hard to distinguish from the reference input. These figures show that by putting more weight on the

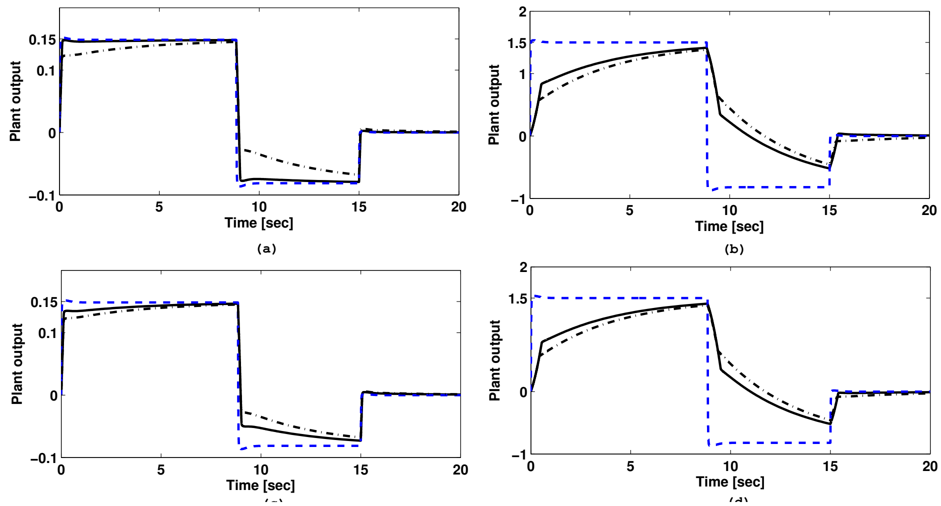

Fig. 3. Output for small (a) and for large (b) reference signal with $c = 200$ and $\tilde{c} = 1$; output for a small (c) and a large (d) reference signal with $c = \tilde{c} = 1$. Solid line is the response of scheduled AW, dashed-dotted line is for traditional AW, and dashed line is for the ideal system.

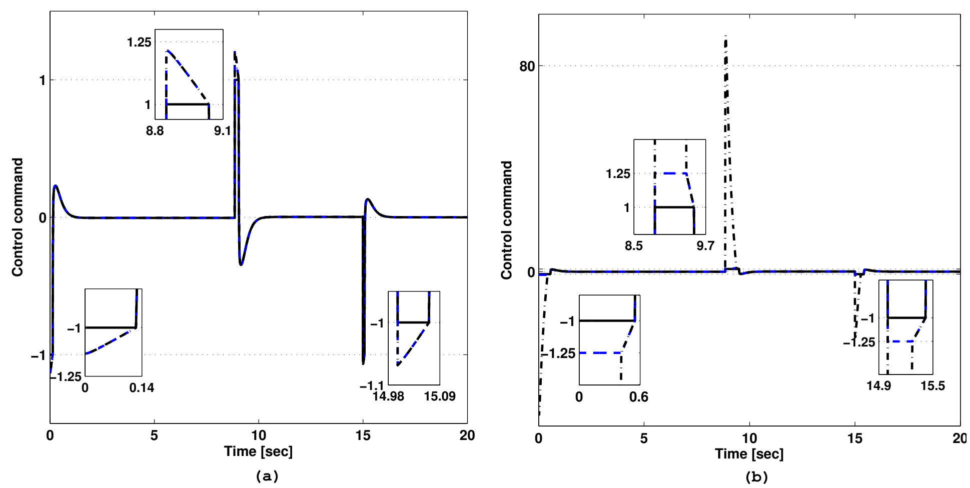

Fig. 4. Control Command for a small (a) and for a large(b) reference signal with $c = 200$ and $\tilde{c} = 1$. Solid line is time history of $\hat{u}$, dashed line is the time history of u_d, and dash-dotted line is the time history of u.

performance of Λ, one can obtain better response for moderate levels of saturation, but at a possible cost of worse overall $\mathcal{L}_2$ gain guarantees. As expected, for high saturation levels, the improvement diminishes.

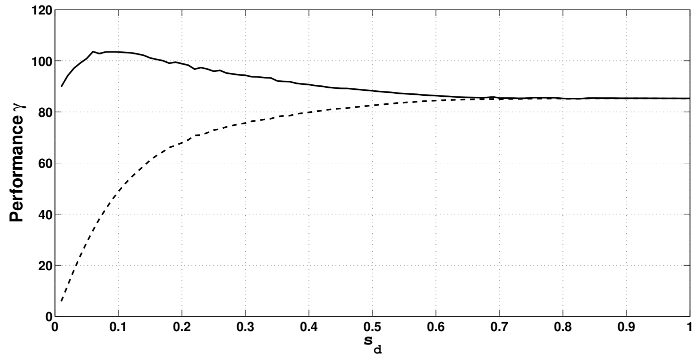

Fig. 5. The $\mathcal{L}_2$ gains $\tilde{\gamma}$ (solid line) and γ (dashed line) verses s_d for $\tilde{c} = c = 1$

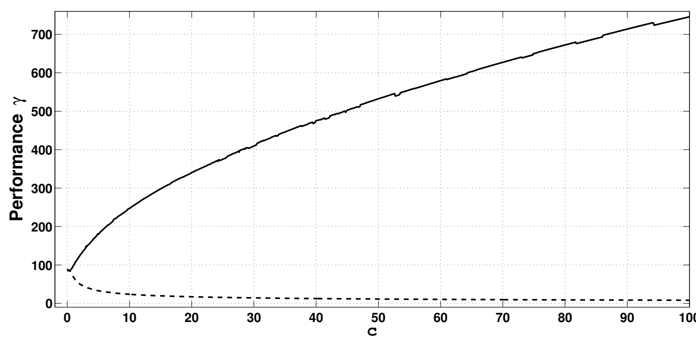

Fig. 6. The $\mathcal{L}_2$ gains $\tilde{\gamma}$ (solid line) and γ (dashed line) verses c with $\tilde{c} = 1$ and $s_d = 0.2$

Figure 5 shows the values of $\tilde{\gamma}$ and γ verses the sector choice s_d. As s_d gets closer to 1, the guaranteed γ becomes larger, since this corresponds to very small values of g_d which essentially removes the first (artificial) saturation element. As a result, $\tilde{\gamma}$ and γ merge and get closer to the value of γ for a single gain. Similarly, small values of s_d corresponds to $g_d \approx 1$ which means that the first (artificial) saturation box limits u_d to the actual saturation limits. This removes Λ and reduces the problem to the case of a single AW loop, thus recovering the results of traditional AW design.

Figure 6 shows the values of $\tilde{\gamma}$ and γ verses the weight c for constant $\tilde{c} = 1$ and $s_d = 0.2$. Again, as one can expect, putting more weight on the design of moderate levels of saturation results in better performance in this region, though this will result in somewhat larger over-all $\mathcal{L}_2$ gain γ.

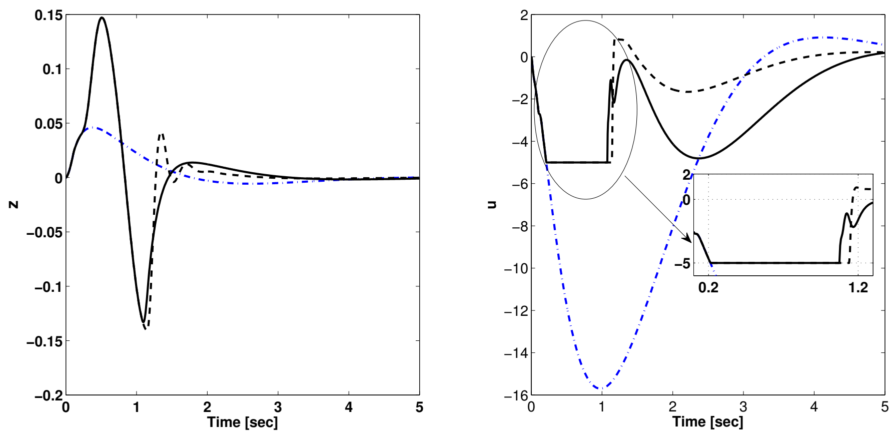

Fig. 7. System Response: nominal ideal system (dashed-dotted), single gain dynamic AW (dashed), scheduled AW (solid)

B. Static-Dynamic AW combination

The following numerical example is taken from [6]. Plant is defined as

$$\left[\begin{array}{c|c} A_p & B_1 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -812.61 & -29.87 & -30.10 & 0 & 15.61 \end{array}\right]$$

$$B_2 = \left[\begin{array}{cccc} 0 & 2.71762 & 0 & 6.68268 \end{array}\right]^T$$

with $z = x_3$, $y_1 = x_1$ and $y_2 = x_3$. We use the same nominal controller as in [6].

The saturation bound is $u_{lim} = 5$. For this example, static AW compensation is not feasible. However, in [6] a plant order dynamic AW is obtained which guarantees a performance level of $\gamma = 181.82$ (the numerical values of the AW matrices are given in [6]). The results of simulation for a step input of duration 0.1 and magnitude 0.5 are depicted in Fig. 7. As this figure shows, the single gain dynamic AW still has some undesirable oscillations. A scheduled combined dynamic and static AW compensator is also designed for this plant with $s_d = 0.2$, $\tilde{c} = 1$ and $c = 20$. This design results in the overall performance of $\tilde{\gamma} = 386.8075$ and inner-loop (moderate levels of saturation) performance of $\gamma = 27.2525$. As Fig. 7 suggests, the scheduled scheme improves the disturbance attenuation behavior of the saturated system by forcing the system to leave the saturation zone earlier than the traditional AW.

VI. Conclusion

We introduced a modified AW scheme with multiple AW gains, along with the convex LMIs to design the gains. The goal is to separate the saturated zone into two parts: moderate levels and higher levels of saturation (extension to multiple regions is possible and is discussed). Then, using the resulting sector conditions, one can design and implement more aggressive anti-windup gains in lower levels of saturation, while global stability and performance requirement are handled by the an outer-loop AW gains, which become active when the system is in higher levels of saturation. We show that similar to the traditional AW, 'all static' AW gains can be infeasible, even if the open-loop plant is open-loop stable. However, by making the outer-loop a dynamic gain, we can end up with an AW structure that open-loop stability is the only feasibility condition. Through two well-known numerical examples, we show that the proposed multiple-stage AW improves the response of the closed-loop system compared to the traditional AW.

Appendix I

System matrices in (7) and (20) are:

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_w & \tilde{B}_q \\ \hline \tilde{C}_z & \tilde{D}_{zw} & D_{zq} \\ \hline \tilde{C}_u & \tilde{D}_{uw} & D_{uq} \end{array}\right] = \left[\begin{array}{cc|c|c} A & B_\eta C_a & B_w & B_q \\ 0 & A_a & 0 & 0 \\ \hline C_z & D_{z\eta} C_a & D_{zw} & D_{zq} \\ \hline C_u & D_{u\eta} C_a & D_{uw} & D_{uq} \end{array}\right]$$

$$\left[\begin{array}{c|c} \tilde{B}_\eta & \tilde{B}_{\tilde{\eta}} \\ \hline D_{z\eta} & D_{z\tilde{\eta}} \\ \hline D_{u\eta} & D_{u\tilde{\eta}} \end{array}\right] = \left[\begin{array}{c|c} B_\eta & B_\eta [0_{(n_u+n_c)\times n_a} \quad I_{n_c+n_u}] \\ 0 & [I_{n_a} \quad 0_{n_a\times(n_c+n_u)}] \\ \hline D_{z\eta} & D_{12} D_{u\tilde{\eta}} \\ \hline D_{u\eta} & D_{u\eta}[0_{(n_u+n_c)\times n_a} \quad I_{n_c+n_u}] \end{array}\right]$$

where using $\Pi = (I_{n_u} - D_{cy} D_{22})^{-1}$:

$$\left[\begin{array}{c} A \\ \hline C_z \\ \hline C_u \end{array}\right] = \left[\begin{array}{cc} A_p + B_2 \Pi D_{cy} C_2 & B_2 \Pi C_c \\ B_{cy}(C_2 + D_{22} \Pi D_{cy} C_2) & A_c + B_{cy} D_{22} \Pi C_c \\ \hline C_1 + D_{12} \Pi D_{cy} C_2 & D_{12} \Pi C_c \\ \hline \Pi D_{cy} C_2 & \Pi C_c \end{array}\right]$$

$$\left[\begin{array}{c} B_w \\ \hline D_{zw} \\ \hline D_{uw} \end{array}\right] = \left[\begin{array}{c} B_1 + B_2 D_{uw} \\ B_{cw} + B_{cy}(D_{22}D_{uw} + D_{21}) \\ \hline D_{11} + D_{12}D_{uw} \\ \hline \Pi D_{cy}D_{21} + \Pi D_{cw} \end{array}\right]$$

$$\left[\begin{array}{c|c} B_q & B_\eta \\ \hline D_{zq} & D_{z\eta} \\ \hline D_{uq} & D_{u\eta} \end{array}\right] = \left[\begin{array}{c|c} -B_2 + B_2 D_{uq} & B_2 D_{u\eta} \\ B_{cy}D_{22}D_{uq} & B_{cy}D_{22}D_{u\eta} + [I_{n_c} \ \ 0_{n_c\times n_u}] \\ \hline D_{12}D_{uq} - D_{12} & D_{12}D_{u\eta} \\ \hline -\Pi D_{cy}D_{22} & \Pi[0_{n_u\times n_c} \quad I_{n_u}] \end{array}\right]$$

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