Modified Anti-windup Compensators for Stable Plants

Solmaz Sajjadi-Kia and Faryar Jabbari

Abstract

We investigate the effects of deferring the activation of anti-windup by allowing actuators to remain in the saturated regime longer, without any assistance. The basic idea is to apply anti-windup when the performance of the saturated system faces substantial degradation. For this, we present a modified anti-windup scheme along with the appropriate LMIs to obtain the gains. For two examples, we show that the modified anti-windup scheme renders better performance than the immediate application of anti-windup.

Index Terms

Input saturation, anti-windup synthesis, finite $\mathcal{L}_2$ gain.

I. Introduction

Designing high performance feedback controllers for linear systems with bounded actuators has been one of the major problems in control for decades, for which ad-hoc but intuitive techniques had been used. However, over the last decade several groups have obtained rigorous stability and performance results for linear systems with input saturation. One common method to deal with input saturation is the so-called *Anti-Windup* (AW) compensation method (e.g., [1]-[5]). Typically, AW is a two-step procedure, in which the original linear controller is designed without considering the input saturation. Since this controller is likely to saturate, the system is

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Authors are with the Mechanical and Aerospace Engineering Department, University of California at Irvine, Irvine, CA 92697, email:ssajjadi@uci.edu, fjabbari@uci.edu

augmented with an AW protection loop to handle saturation. In this paper, we provide a modified form of the traditional AW for application to linear systems subject to saturation.

In recent years, the structure shown in Fig. 1, where sat$(\cdot)$ represents the standard decentralized saturation nonlinearity, has been adapted for many AW augmentation schemes. Traditionally, properties of AW schemes, such as stability guarantees and graceful degradation of performance of systems were addressed through extensive simulations. More recently, by relying on numerical solvers for Linear Matrix Inequalities (LMIs), stability and performance guarantees have been developed for cases where the augmentation considered is static or dynamic with an order matching that of the plant (e.g., [2], [3], respectively).

In this paper, we consider the static AW augmentation synthesis problem. For this, we use an approach based on the Small Gain Theorem (SGT). The results lead to a variation of the LMIs typically used for AW synthesis (e.g., those in [2] and [3]), though the two approaches are closely related (e.g. see Lemma 3.5.6 in [6]). The main contribution of this paper, developed in Section III, is investigating the effect of postponing the activation of AW. The rationale behind this intentional delay is a tradeoff between the two possible modes of operation. In one, AW is active as soon as saturation is encountered, resulting in a stable but typically low performance controller. On the other hand, if the actuator command is slightly or moderately above saturation, the nominal controller acts as a high performance controller subjected to a modest amount of parameter uncertainty at the input. The basic idea is not to apply AW action as soon as saturation is encountered, but instead allow saturated actuators act unassisted up to a point, to be made precise below. This is based on the assumption that the nominal controller possesses a reasonable amount of performance robustness. This idea is somewhat related to the over-saturation (e.g., [7]) or high-gain (e.g., [8], [9]) approaches that have been used in the direct and explicit approach to saturation though, unlike these references, the controller used in the small signal regime here is the high performance nominal one which is augmented in a manner very similar to AW schemes. The results are studied through two examples. The preliminary results of this paper were presented in [10]. Here a complete version with detailed description for the multi-input systems has been provided, along with a new example. Notations are quite standard, though to avoid confusion certain concepts and notations are defined or discussed, as the need arises. To reduce clutter, off-diagonal entries in symmetric matrices are replaced with '$\star$'.

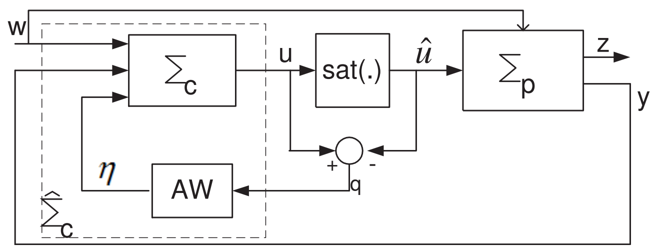

Fig. 1. Standard anti-windup augmentation scheme

II. Static Anti-windup Synthesis

A. Problem Definition

Consider a system with a *nominal* controller designed to fulfill a specific task, such as tracking or disturbance regulation for which, due to possible saturation, an AW block is needed. Consider Fig. 1, in which

$$\Sigma_p \quad \sim \quad \begin{cases} \dot{x}_p &= A_p x_p + B_1 w + B_2 \hat{u} \\ y &= C_2 x_p + D_{21} w + D_{22} \hat{u} \\ z &= C_1 x_p + D_{11} w + D_{12} \hat{u} \end{cases} \tag{1}$$

is the the plant dynamics with $x_p \in \mathbb{R}^n$ the plant state, $\hat{u} \in \mathbb{R}^{n_u}$ the control input, $w \in \mathbb{R}^{n_w}$ the exogenous input, $y \in \mathbb{R}^{n_y}$ the measurement output, and $z \in \mathbb{R}^{n_z}$ the controlled output. The plant with $\hat{u} \equiv 0$ is the *open-loop* plant. The controller is described as

$$\widehat{\Sigma}_c \quad \sim \quad \begin{cases} \dot{x}_c &= A_c x_c + B_{cy} y + B_{cw} w + \eta_1 \\ u &= C_c x_c + D_{cy} y + D_{cw} w + \eta_2 \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ and $u \in \mathbb{R}^{n_u}$ are the controller state and output vectors, respectively, while η_1 and η_2 are inputs due to the AW. When $\eta_1 = 0$ and $\eta_2 = 0$, then $\widehat{\Sigma}_c$ becomes Σ_c, the nominal controller. The saturation is assumed decentralized with the limit u_{lim} for each u_i ($i = 1, 2, ..., n_u$):

$$\hat{u}_i = sat(u_i) = sgn(u_i)\, min\{|u_i|, u_{lim}\}$$

The purpose of the AW block in Fig. 1 is to introduce correction terms in the controller to counteract adverse effects of saturation. Ideally, these correction terms should not affect the control loop as long as actuators do not saturate. A common choice for AW block is static

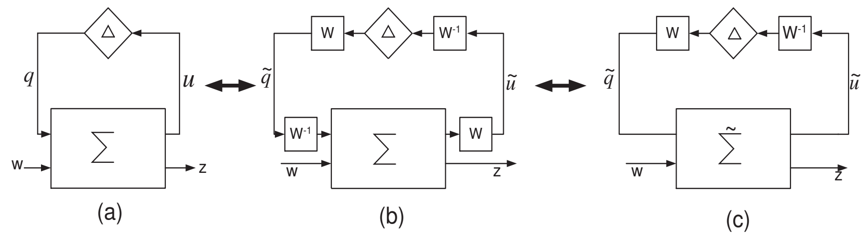

Fig. 2. Standard interconnection for closed-loop system

constant gains (see [2]) where, for $q = u - \hat{u}$, the AW block in Fig. 1 is of the structure

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = -\Lambda q = -\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} q. \tag{3}$$

B. LMI-Based Anti-windup Synthesis

Often, saturation function (or deadzone function) is treated as a sector nonlinearity and the problem is cast in the general framework of absolute stability. Then, tools such as the Circle criterion, mostly with LMI characterization developed with quadratic Lyapunove/storage functions, are used to design stabilizing AW gains (see [2] and [3] for two well-known representatives). Here, we use the scaled Small Gain Theorem (SGT) approach which results in a variation of the the LMIs obtained in [2] and [3]. Originally, use of SGT was motivated by the potential for incorporating a measure of scheduling in the basic AW scheme. For the purposes of this paper however, either approach would have sufficed and provided essentially the same results. In all of these methods including this work, since we ensure stability for any possible saturation level, we are restricting ourselves to open-loop stable plants Σ_p.

By selecting $x = [x_p^T \ \ x_c^T]^T$, and considering w and $q = u - \hat{u}$ as input, we can represent the closed-loop system (1), (2), and (3) in the LFT format shown in Fig. 2(a) where Δ is a diagonal matrix with each entry $1(\cdot) - sat(\cdot)$. To reduce conservatism, we introduce the diagonal $n_u \times n_u$ scaling matrix $W > 0$, as shown in Fig. 2(b), where W (technically $M = W^{-2}$) becomes a search variable. Then,

$$\tilde{\Sigma} \ \sim \ \begin{cases} \dot{x} = Ax + B_w w + (B_q - B_\eta \Lambda) W^{-1} \tilde{q} \\ z = C_z x + D_{zw} w + (D_{zq} - D_{z\eta}\Lambda) W^{-1} \tilde{q} \\ \tilde{u} = W C_u x + W D_{uw} w + W(D_{uq} - D_{u\eta}\Lambda) W^{-1} \tilde{q} \end{cases} \tag{4}$$

where A, B_η, etc. are determined by the matrices in (1) and (2). For decentralized saturation, Δ is a diagonal dead-zone matrix, thus $\|\Delta\|_2 \leq 1$. This, and diagonal $W > 0$, imply $\|W\Delta W^{-1}\|_2 \leq 1$. Therefore, to have the feedback system in Fig. 2(c) stable, the SGT condition becomes:

$$\|\tilde{\Sigma}_{\tilde{u}\tilde{q}}\|_{2,i} < 1. \tag{5}$$

To establish a performance bound, typically $\mathcal{L}_2$ gain from w to z is considered. Following the standard approach, by relying on the quadratic Lyapunov function $V = x^T Q^{-1} x$ with $Q > 0$, the $\mathcal{L}_2$ gain from w to z, γ, can be obtained using the standard inequality

$$\frac{d}{dt}(x^T Q^{-1} x) + \gamma^{-1} z^T z - \gamma w^T w < 0. \tag{6}$$

Since $\|W\Delta W^{-1}\|_2 \leq 1$, we have $\tilde{q}^T\tilde{q} - \tilde{u}^T\tilde{u} \leq 0$. Then, invoking the S-procedure for some $\tau > 0$,

$$\frac{d}{dt}(x^T Q^{-1} x) + \gamma^{-1} z^T z - \gamma w^T w - \tau(\tilde{q}^T\tilde{q} - \tilde{u}^T\tilde{u}) < 0 \tag{7}$$

is the sufficient condition for inequality (6). Expanding this inequality and preforming proper congruent transformations, inequality (7) can be written in the LMI form of the theorem below with $M = \frac{1}{\tau}W^{-2}$ and $X = \Lambda M$.

Theorem 1 (Synthesis and performance)**.** *The closed loop system in Fig. 2 is stable and the $\mathcal{L}_2$ gain from w to z is less than γ, if there exist diagonal matrix $\mathbf{M} > 0$, symmetric matrix $\mathbf{Q} > 0$, and matrix $\mathbf{X}$ satisfying*

$$\begin{pmatrix} A\mathbf{Q} + \mathbf{Q}A^T & \star & \star & \star & \star \\ \mathbf{M}B_q^T - \mathbf{X}^T B_\eta^T & -\mathbf{M} & \star & \star & \star \\ C_u\mathbf{Q} & D_{uq}\mathbf{M} - D_{u\eta}\mathbf{X} & -\mathbf{M} & \star & \star \\ B_w^T & 0 & D_{uw}^T & -\gamma I & \star \\ C_z\mathbf{Q} & D_{zq}\mathbf{M} - D_{z\eta}\mathbf{X} & 0 & D_{zw} & -\gamma I \end{pmatrix} < 0. \tag{8}$$

If the LMI is feasible, then the AW gain can be obtained from $\Lambda = \mathbf{X}\mathbf{M}^{-1}$.

Note that the block (1:3,1:3) is the sufficient LMI condition for (5), thus stability is ensured. The well-posedness of the closed-loop system under the AW augmentation is guaranteed by the block (2:3,2:3) (see the Appendix for details.)

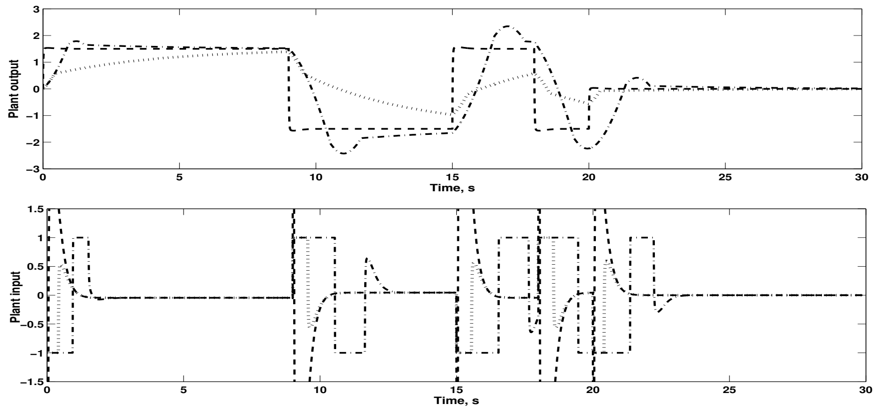

Fig. 3. Plant output and input plots: unconstrained ideal system (dashed), saturated nominal closed-loop with no AW (dashed-dotted), saturated closed-loop with AW (dotted).

C. Motivating Numerical Example

Consider the following system taken from [4] with input bound $u_{lim} = 1$,

$$\left[\begin{array}{c|c|c} A_p & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \end{array}\right] = \left[\begin{array}{ccc|c|c} -10.6 & -6.09 & -0.9 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c|c} A_c & B_{cy} & B_{cr} \\ \hline C_c & D_{cy} & D_{cr} \end{array}\right] = \left[\begin{array}{cc|c|c} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array}\right]$$

with $z = y - r$, where r is the reference signal. The nominal closed loop γ is 1. Using Theorem 1, the AW gain is $\Lambda = [-0.1968 \quad 0.0025 \quad -0.9860]^T$, with a performance level of $\gamma = 85.78$, very close to the result obtained in [4] for the static AW, which was $\gamma = 86.07$. The slight difference probably is the result of different options used in the LMI solver. Figure 3 shows the response with and without the AW augmentation for the the reference signal used in [4]. The response is essentially the same as the one obtained in [4]. Here, by ‘Constrained Nominal’, we mean the system with bounded actuator(s), under nominal controller but without the AW.

III. A Modified Anti-windup Scheme

Figure 3 suggests that for this example, constrained nominal closed-loop system (dash-dotted) shows better tracking behavior than the system with AW (solid), especially the first 10 seconds. A possible explanation might be the tradeoff between the two possible modes of operation mentioned before: in one, AW is active as soon as saturation is encountered, resulting in a safe (i.e., stable) but a low performance controller, in the sense that typically the only performance guarantees are no better than the open-loop. Recall that the nominal γ was 1, while γ of the AW was 85.78. On the other hand, if the actuator command is only moderately above saturation, the nominal controller acts as a high performance controller subjected to a modest amount of parameter uncertainty at the input, or 'matched' uncertainty, which is a mild form of uncertainty.

Based on this observation, the main idea here is to postpone the activation of AW to a point where system really needs AW protection. To investigate further, we first consider the performance of the constrained system without AW as a function of the maximum value of the command sent to the saturation box. For notational simplicity, we first discuss the single input case. The generalization to multi-input systems will be discussed later.

A. Single-input Plants

The nonlinear saturation element with input $u(t)$, output $\hat{u}(t)$, and saturation bound u_{lim} can be replaced by the time varying gain $G(t)$:

$$\hat{u}(t) = G(t)u(t), \quad G(t) = \begin{cases} 1 & |u(t)| \leq u_{lim} \\ sgn(u(t))\frac{u_{lim}}{u(t)} & |u(t)| > u_{lim} \end{cases}. \tag{9}$$

The idea of modeling saturation with a time varying gain has been exploited before (e.g., see [9],[11], and [12].) When the actuator is not saturated, $G = 1$. If $|u(t)|$ can be bounded with $\frac{1}{g}u_{lim}$, for $0 \leq g < 1$, then the minimum value retained by $G(t)$ is g, i.e., $G(t) \in [\,g\,,1]$. For the rest of this paper, we assume $D_{22} = 0$. Then, considering (9), the nominal constrained closed-loop system (equations (1),(2)) with $x = [x_p^T \quad x_c^T]^T$ can be written as

$$\begin{cases} \dot{x} = A_{cl}(G(t))x + B_{cl}(G(t))w \\ z = C_{cl}(G(t))x + D_{cl}(G(t))w \end{cases} \tag{10}$$

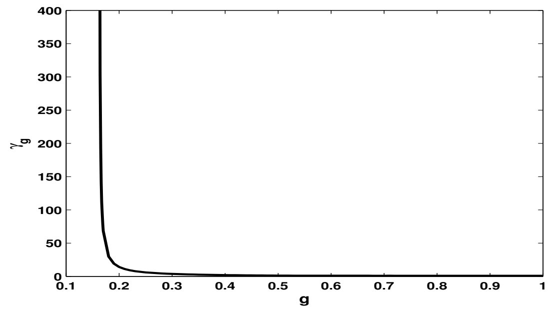

Fig. 4. Performance (i.e., $\mathcal{L}_2$ gain) of the saturated system for different level of saturation, i.e, different values of g in $G(t) \in [\,g\,,\,1]$.

with

$$\left[\begin{array}{c} A_{cl}(G(t)) \\ \hline C_{cl}(G(t)) \end{array}\right] = \left[\begin{array}{cc} A_p + B_2 G(t) D_{cy} C_2 & B_2 G(t) C_c \\ B_{cy} C_2 & A_c \\ \hline C_1 + D_{12} G(t) D_{cy} C_2 & D_{12} G(t) C_c \end{array}\right]$$

$$\left[\begin{array}{c} B_{cl}(G(t)) \\ \hline D_{cl}(G(t)) \end{array}\right] = \left[\begin{array}{c} B_1 + B_2 G(t) D_{cy} D_{21} + B_2 G(t) D_{cw} \\ B_{cw} + B_{cy} D_{21} \\ \hline D_{11} + D_{12} G(t) D_{cy} D_{21} + D_{12} G(t) D_{cw} \end{array}\right].$$

The matrices above represent an LPV system with variable $G(t) \in [\,g\,,\,1]$ $(0 \le g)$. Note that, $G(t)$ is actually $G(u(t))$ and, since $u(t)$ is dependent on states, the system has a quasi-LPV form. Here, we use g as a measure of the level of saturation. To study the performance of the saturated nominal closed-loop system for different levels of saturation, we *assume that* $|u(t)| \le \frac{1}{g} u_{lim}$ for all t. Then, an estimate for the $\mathcal{L}_2$ gain of this closed-loop can be obtained from minimizing γ_g subject to $\mathbf{Q} > 0$ and

$$\begin{pmatrix} \mathbf{Q} A_{cl}(\bar{g})^T + A_{cl}(\bar{g}) \mathbf{Q} & \star & \star \\ B_{cl}(\bar{g})^T & -\gamma_g I & \star \\ C_{cl}(\bar{g}) \mathbf{Q} & D_{cl}(\bar{g}) & -\gamma_g I \end{pmatrix} < 0 \quad \text{for } \bar{g} = \{g\,,\,1\} \tag{11}$$

where matrix $A_{cl}(\bar{g})$, etc mean value of $A_{cl}(G(t))$ evaluated at $G(t) = \bar{g}$.

Figure 4 shows the result of analysis for the example of Section II-C. In this figure, each point (g, γ) on the curve represents the performance of the system for $G(t) \in [\,g\,,\,1]$, i.e. if

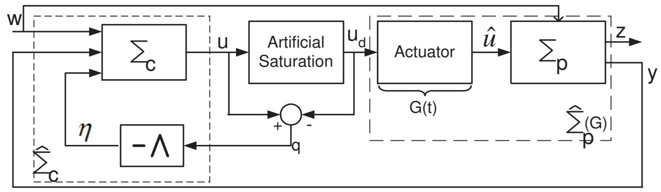

Fig. 5. New modified anti-windup scheme. Note that $u_d \in [\frac{-u_{lim}}{g_d}, \frac{u_{lim}}{g_d}]$ and $\hat{u} \in [-u_{lim}, u_{lim}]$

the actuator is guaranteed to receive – *somehow* – control command with peak value of $\frac{1}{g}u_{lim}$ or less. Figure 4 suggests that such a constrained nominal closed-loop system has adequate performance up to about $g = 0.2$, or equivalently $|u(t)| \leq \frac{1}{0.2}1 = 5$ (i.e., $G(t) \in [0.2\,,\,1]$), without any assistance from any external compensator. Of course, in general and without further modification, we cannot know or limit the peak value of the command to the actuator.

Motivated by this, we propose the scheme shown in Fig. 5 for the deferral of AW activation. In this new scheme, we add an artificial saturation element with saturation bounds $\pm\frac{1}{g_d}u_{lim}$, where g_d is the design variable, specified by designer, thus $u_d(t) = sgn(u(t))\ min\{(|u(t)|, \frac{1}{g_d}u_{lim}\}$. Then, we use $q(t) = u(t) - u_d(t)$, the difference between the input and output of the artificial saturation (implemented easily via software), as the input signal to the static AW block. This lets the saturated system stay unassisted up to the point g_d, i.e., AW activates only when $|u(t)| > \frac{1}{g_d}u_{lim}$. One can obtain the design point, g_d, by trial and error, focused around the 'bend' in the γ_g vs. g plots similar to the one in Fig. 4. Note that g_d depends on how much input uncertainty the nominal closed-loop can tolerate. A desirable nominal controller can be expected to have a reasonable amount of performance robustness, particularly to matched uncertainty in the B_2 matrix, allowing a relatively small g_d.

In Fig. 5, since the input to the actuator satisfies $|u_d(t)| \leq \frac{1}{g_d}u_{lim}$, we have $\hat{u}(t) = G(t)u_d(t)$ where $G(t) \in [g_d, 1]$, and if $|u_d(t)| > \frac{1}{g_d}u_{lim}$, then $G(t) = g_d$. Thus, instead of (1)-(3), we have

$$\begin{cases} \dot{x}_p &= A_p x_p + B_1 d + B_2 G(t) u_d \\ y &= C_2 x_p + D_{21} w \\ z &= C_1 x_p + D_{11} w + D_{12} G(t) u_d \end{cases}$$

$$\begin{cases} \dot{x}_c &= A_c x_x + B_{cy} y + B_{cw} w - [I \quad 0]\Lambda q \\ u &= C_c x_c + D_{cy} y + D_{cw} w - [0 \quad I]\Lambda q \end{cases}.$$

The structure remains the same as the original problem except that now the control input matrices, B_2 and D_{12}, are effectively $B_2G(t)$ and $D_{12}G(t)$. We use a diagonal $W > 0$ for scaling as before, i.e., we have $\tilde{u} = Wu$ and $\tilde{q} = Wq$. Then, we can depict Fig. 5 in the similar LFT format of Fig. 2. The only difference is that now the system matrices of Σ and $\widetilde{\tilde{\Sigma}}$ have $G(t)$ which appears linearly; thus, we only need to set $D_{22} = 0$, and replace B_2 and D_{12} with $B_2G(t)$ and $D_{12}G(t)$, respectively, as in

$$A(G(t)) = \begin{bmatrix} A_p + B_2G(t)D_{cy}C_2 & B_2G(t)C_c \\ B_{cy}C_2 & A_c \end{bmatrix}. \tag{12}$$

Since the saturation level does not enter the design inequalities that were in Theorem 1, one can follow the steps used in Theorem 1 by using the same $V = x^TQ^{-1}x$ and the $\mathcal{L}_2$ inequality in (6)-(7), except for a plant that has a parameter varying B_2 and D_{12}. Due to linearity, this leads to two matrix inequalities of the form in (8), one for each extreme value of $G(t)$: 1 or g_d. However, the interaction between the two saturation elements can be used to reduce the dimension of one of the inequalities: Consider the modified AW scheme in Fig. 5. For the Lyapunov function $V = x^TQ^{-1}x$, (6) ensures that the $\mathcal{L}_2$ gain of the closed-loop system under the modified AW is less than γ. Depending on the condition of the artificial saturation element, the system is under one of the following two operating modes:

- When $|u(t)| \leq \frac{1}{g_d}u_{lim}$, i.e., the artificial saturation element is not saturated; we have $q = 0$ ($\tilde{q}(t) = 0$). Therefore, inequality (6), applied for this case, can be written in the LMI form

$$\begin{pmatrix} A(G(t))\mathbf{Q} + \mathbf{Q}A(G(t))^T & \star & \star \\ B_w(G(t))^T & -\gamma I & \\ C_z(G(t))^T\mathbf{Q} & D_{zw}(G(t)) & -\gamma I \end{pmatrix} < 0 \tag{13}$$

where variable $G(t) \in [g_d, 1]$. It is sufficient to check this inequity for $G = g_d$ and $G = 1$.

- When $|u(t)| > \frac{1}{g_d}u_{lim}$, we have $|u_d(t)| = \frac{1}{g_d}u_{lim}$. As a result $q(t) \neq 0$ ($\tilde{q}(t) \neq 0$), but $\hat{u}(t) = g_du_d(t)$, i.e., $G(t)$ is now a constant gain. Following the steps used to get (8), we get the LMI (15) in the theorem below.

Thus at any given time, depending on the operating condition of the artificial saturation element, either (14) or (15) below ensures (6) holds for the modified AW scheme, which establishes stability and $\mathcal{L}_2$ gain.

Theorem 2 (Single input)**.** *The closed loop system in Fig. 5 is stable and the $\mathcal{L}_2$ gain from w to z is less than γ, if there exist a scalar $\mathbf{M} > 0$, symmetric matrix $\mathbf{Q} > 0$, and matrix $\mathbf{X}$ satisfying*

$$\begin{pmatrix} A(\bar{g})\mathbf{Q} + \mathbf{Q}A(\bar{g})^T & \star & \star \\ B_w(\bar{g})^T & -\gamma I & \star \\ C_z(\bar{g})\mathbf{Q} & D_{zw}(\bar{g}) & -\gamma I \end{pmatrix} < 0 \text{ for } \bar{g} = 1 \tag{14}$$

$$\begin{pmatrix} \Omega_{1\times1}(g_d) & \star & \star & \star & \star \\ B_w(g_d)^T & -\gamma I & \star & \star & \star \\ C_z(g_d)\mathbf{Q} & D_{zw}(g_d) & -\gamma I & \star & \star \\ \Omega_{4\times1}(g_d) & 0 & \Omega_{4\times3}(g_d) & -\mathbf{M} & \star \\ C_u\mathbf{Q} & D_{uw} & 0 & \Omega_{5\times4}(g_d) & -\mathbf{M} \end{pmatrix} < 0 \tag{15}$$

with

$$\begin{cases} \Omega_{1\times1}(g_d) = A(g_d)\mathbf{Q} + \mathbf{Q}A(g_d)^T \\ \Omega_{4\times1}(g_d) = \mathbf{M}B_q(g_d)^T - \mathbf{X}^T B_\eta(g_d)^T \\ \Omega_{4\times3}(g_d) = \mathbf{M}D_{zq}(g_d)^T - \mathbf{X}^T D_{z\eta}(g_d)^T \\ \Omega_{5\times4}(g_d) = D_{uq}\mathbf{M} - D_{u\eta}\mathbf{X} \end{cases}$$

where $A(g_d)$, etc mean value of $A(G(t))$ in (12) evaluated at $G(t) = g_d$. If this problem is feasible, then the AW gain, Λ, can be obtained from $\Lambda = XM^{-1}$.

LMI (14) holding for both $\bar{g} = 1$ and $\bar{g} = g_d$ is a sufficient condition for LMI (13). However, note that the block (1:3,1:3) of LMI (15) is the repetition of LMI (14) for $\bar{g} = g_d$. Therefore, we do not need LMI (14) for $\bar{g} = g_d$. Also, (15) is exactly the same as (8) except some rows/colums are reordered (to see the connection to (14) more clearly). One could have used two inequalities of the form in (8), one evaluated at $\bar{g} = 1$ and one at $\bar{g} = g_d$. The form above reduces size of one of the inequalities (as in (14)). While this might be of marginal benefit in a single input case, the benefits grow for the multi-input case, which are discussed next.

B. Multi-input Plants

In case of multiple actuators, we use the same modified scheme of Fig. (5) with the same approach as in the single actuator case to model the system under the modified AW. The only difference is that now the variable $G(t)$, equivalent gain for actuators saturation box, is a $n_u \times n_u$ diagonal matrix with each entry similar to (11), i.e.,

$$G(t) = diagonal\{G_i(t)\} \quad \text{with} \quad G_i(t) \in [g_{d_i}, 1]$$

where $0 \leq g_{d_i} < 1, \ i = 1, 2, ..., n_u$, is the design point chosen for the i^{th} actuator. Given linearity, any $G(t)$ can be represented as a linear combination of extreme values, evaluated at the corners of the parameter hypercube. So, $G(t) = \sum_{k=1}^{2^{n_u}} \alpha_k(t)\, \overline{G}^k$ ($\sum_{k=1}^{2^{n_u}} \alpha_k(t) = 1$), where $\forall\, i = 1, 2, \cdots, n_u$

$$\overline{G}^k \in \mathcal{G}, \quad k = 1, 2, \cdots, 2^{n_u}, \quad \mathcal{G} = \{G(t) : G_i(t) = g_{d_i} \text{ or } 1\} \tag{16}$$

Then, the sufficient condition to establish stability and an estimate for the $\mathcal{L}_2$ gain of the closed-loop system is simply checking the analog of the the inequality (8) at the vertices obtained from $G(t)$ (i.e., $\overline{G}^k, \quad k = 1, 2, \cdots, 2^{n_u}$).

Theorem 3 (Multi-input)**.** *The closed loop system in Fig. 5 is stable and the $\mathcal{L}_2$ gain from w to z is less than γ, if there exist an $n_u \times n_u$ diagonal matrix $\mathbf{M} > 0$, symmetric matrix $\mathbf{Q} > 0$, and matrix $\mathbf{X}$ satisfying*

$$\Phi^k < 0 \quad \text{for} \quad k = 1, 2, ..., 2^{n_u} \tag{17}$$

where $\Phi^k =$

$$\begin{pmatrix} \Phi_{1\times1}(k) & \star & \star & \star & \star \\ B_w(\overline{G}^k)^T & -\gamma I & \star & \star & \star \\ C_z(\overline{G}^k)\mathbf{Q} & D_{zw}(\overline{G}^k) & -\gamma I & \star & \star \\ \Phi_{4\times1}(k) & 0 & \Phi_{4\times3}(k) & -\mathbf{M} & \star \\ C_u\mathbf{Q} & D_{uw} & 0 & D_{uq}\mathbf{M} - D_{u\eta}\mathbf{X} & -\mathbf{M} \end{pmatrix} \tag{18}$$

$\Phi_{1\times1}(k) = A(\overline{G}^k)\mathbf{Q} + \mathbf{Q}A(\overline{G}^k)^T$, $\Phi_{4\times1}(k) = \mathbf{M}B_q(\overline{G}^k)^T - \mathbf{X}^T B_\eta(\overline{G}^k)^T$, $\Phi_{4\times3}(k) = \mathbf{M}D_{zq}(\overline{G}^k)^T - \mathbf{X}^T D_{z\eta}(\overline{G}^k)^T$, *and* $\overline{G}^k$ *is defined in (16). If this set is feasible, then the modified AW gain, Λ, can be obtained from $\Lambda = \mathbf{X}\mathbf{M}^{-1}$.*

While in a typical uncertain (or LPV) problem satisfaction at the corners of the parameter hypercube is necessary, similar to the development of Section (III-A), the structure of Fig. 5 can be used to reduce the size of some of the matrix inequalities in Theorem 3.

For this, one can methodically apply (6) for the 2^{n_u} possible operating conditions in which a subset of artificial actuators are saturated, starting with all saturated, then any one unsaturated, then any combination of two unsaturated, etc. For a given operating condition, if any of the artificial saturation elements, say i^{th} one, is saturated, we have $G_i(t) = g_{d_i}$, and when it is not saturated we have $q_i(t) = 0$ and $G_i(t) \in [1\,,\, g_{d_i}]$. In the first case (all saturated), we get an LMI similar to (18) with $G_i(t) = g_{d_i}\ \forall i$. In the operating conditions where one is unsaturated, similar to Section III-A (LMI (13)), the AW gains for the unsaturated actuator do not enter the resulting LMI, leading to a smaller LMI (fewer rows/columns). However, in the unsaturated channel $G_i(t) \in [1\,,\, g_{d_i}]$, resulting in an LMI with time varying element (similar to (13)) for each any-one-unsaturated case. A sufficient condition for every LMI is to evaluate it at $G_i(t)$'s corners: 1 and g_{d_i}. However, the LMI corresponding to g_{d_i} corner is a sub-block of the LMI of the operating condition in which that actuator is saturated (in this case, the LMI with $G_i(t) = g_{d_i}\ \forall i$, seen in the very first case), and thus is automatically satisfied. Hence, we only keep the LMI corresponding to the corner $G_i(t) = 1\ \forall i$. This procedure is repeated for the rest of the operating conditions. In each operating condition, we get only one new LMI in which the entries of $\overline{G}^k$ corresponding to unsaturated actuators are 1 and the rows/columns of the AW gains of those actuators are eliminated for that LMI. The resulting reduced dimension LMIs are summarized in Theorem 4 below. First, we need to introduce matrices $\mathcal{T}^k$: For each diagonal $\overline{G}^k$, we define a $\mathcal{T}^k$ as the $n_u \times n_u$ identity matrix in which, for $i = 1, 2, \cdots, n_u$ the i^{th} row is removed if $\overline{G}_i^k = 1$.

Theorem 4 (Multi-input, reduced size)**.** *The closed loop system in Fig. 5 is stable and the $\mathcal{L}_2$ gain from w to z is less than γ, if there exist an $n_u \times n_u$ diagonal matrix $\mathbf{M} > 0$, symmetric matrix $\mathbf{Q} > 0$, and matrix $\mathbf{X}$ satisfying*

$$\Psi^k < 0 \quad \text{for} \quad k = 1, 2, ..., 2^{n_u} \tag{19}$$

where $\Psi^k =$

$$\begin{pmatrix} \Psi_{1\times1}(k) & \star & \star & \star & \star \\ B_w(\overline{G}^k)^T & -\gamma I & \star & \star & \star \\ C_z(\overline{G}^k)\mathbf{Q} & D_{zw}(\overline{G}^k) & -\gamma I & \star & \star \\ \Psi_{4\times1}(k) & 0 & \Psi_{4\times3}(k) & -\mathcal{T}^k\mathbf{M}\mathcal{T}^{k^T} & \star \\ \mathcal{T}^k C_u\mathbf{Q} & \mathcal{T}^k D_{uw} & 0 & \Psi_{5\times4}(k) & -\mathcal{T}^k\mathbf{M}\mathcal{T}^{k^T} \end{pmatrix} \tag{20}$$

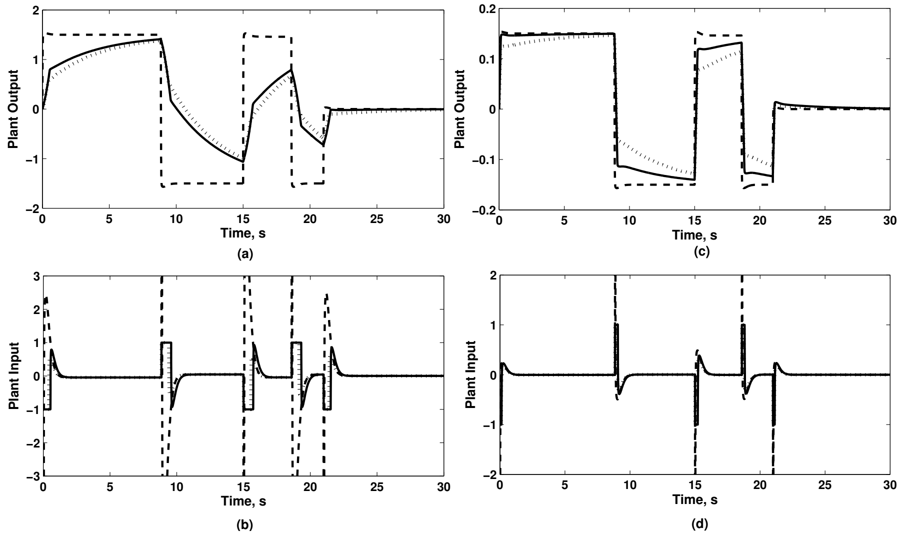

Fig. 6. Plant output and input for the example of Section II-C: Unconstrained ideal response (dashed), immediate activation of AW (dotted) and modified AW (solid). (a), (b) the same reference signal of Section II-C (peak value 1.5) (c), (d) smaller reference signal (peak value 0.15).

$\Psi_{1\times1}(k) = A(\overline{G}^k)\mathbf{Q} + \mathbf{Q}A(\overline{G}^k)^T$, $\Psi_{4\times1}(k) = \mathcal{T}^k(\mathbf{M}B_q(\overline{G}^k)^T - \mathbf{X}^T B_\eta(\overline{G}^k)^T)$, $\Psi_{4\times3}(k) = \mathcal{T}^k(\mathbf{M}D_{zq}(\overline{G}^k)^T - \mathbf{X}^T D_{z\eta}(\overline{G}^k)^T)$, $\Psi_{5\times4}(k) = \mathcal{T}^k(D_{uq}\mathbf{M} - D_{u\eta}\mathbf{X})\mathcal{T}^{k^T}$ and $\overline{G}^k$ defined in (16). If this set is feasible, then the modified AW gain, Λ, can be obtained from $\Lambda = \mathbf{XM}^{-1}$.

C. Numerical Examples

Continuing with the example of Section II-C: Using Theorem 2 with design point $g_d = 0.17$, $\Lambda = [-0.003 \;\; 0 \;\; -1]^T$ is obtained for the modified AW. This compensator has $\gamma = 87.50$. With $g_d = 1$, i.e, immediate activation of AW, we had $\gamma = 85.78$. The overall γ for the modified scheme is slightly higher than that of the traditional one, as expected (due to uncertainty in control input matrix). As Fig. 6 suggests, however, we get better results with the modified scheme (solid line), especially when the system is slightly saturated, particulary for smaller reference (Fig. 6.c).

Example 2: Consider the forth order lateral dynamics of F16 aircraft with states $[\beta \;\; \phi \;\; p \;\; r]^T$, and control inputs: aileron deflection and rudder deflection, augmented with the actuator dynamics for each channel in the form of $\frac{20.2}{s+20.2}$, given in [13] page 577. The following LQG controller with $y = [\phi \;\; \beta]^T$ is designed to maintain a desired bank angle ϕ_r for the aircraft:

$C(s) = \frac{1}{s}P(-K\,[sI - A_p - B_2K - LC_2]^{-1}\,L)$. The reference [13] studies the response of the aircraft for given predesigned values of the matrices L and P, and different values of K obtained using the state weight $Q = C_2^T C_2$ and different values of control input weight R. Here we use a K obtained from $R = \mathrm{diagonal}\begin{bmatrix} 10^{-11} & 10^{-8} \end{bmatrix}$. The saturation limit of each control surface is ± 15 degree. The controlled output is $z = \phi - \phi_r$. Figure 7 illustrates the bank angle of the aircraft ϕ, and the control inputs to the aircraft for a command of $\phi_r = 6°$. The control inputs to the aircraft are both almost zero after $t = 0.6\ sec$, thus, for the clarity of illustration of the early parts, the time history of the inputs are only shown for $t = [0\quad 1]$. For the given command, as the input to the system plots show, in the constrained nominal case, i.e. no AW (dash-dotted), both actuators are saturated resulting in a big overshoot in the ϕ response compared to the the ideal unconstrained closed-loop response (dashed).

Using the traditional AW scheme (procedure of Section II-B), a static AW compensator can be obtained, which guarantees a $\mathcal{L}_2$ gain of $\gamma = 85.9$. Next, we use Theorem 4, to design a modified AW. Using plots similar to Fig. 4, after some trial and error, the following design point is chosen: $g_{d_1} = 0.8,\ g_{d_2} = 0.25$. Using this design point, a modified AW compensator is designed whose overall $\mathcal{L}_2$ gain is $\gamma = 181.37$. Figure 7 also shows the response of the system with the immediate activation of the AW (dotted), and response of system under the modified AW (solid). The goal of the modified AW is to generate a response closer to the ideal unconstrained closed-loop system; as Fig 7 shows the system with the modified AW accomplishes this goal.

In the case of multiple actuators interpretation of the results are not as straightforward as the single actuator case because of the coupling induced by the static AW gains in the controller channels. For example, here, both actuators are saturated in the constrained nominal case, however, in both traditional and modified AW cases the first actuator does not saturated; this can be as a result of the early activation of the AW due to the second actuator's saturation. Although γ for the modified AW is higher than the traditional case, as expected, the system has better response. Closer examination of the output plot shows that the constrained nominal system is faster than the system with the traditional AW. The deferral of the activation of the AW lets the system take advantage of the this fast response, and generate a response closer to the ideal nominal closed-loop.

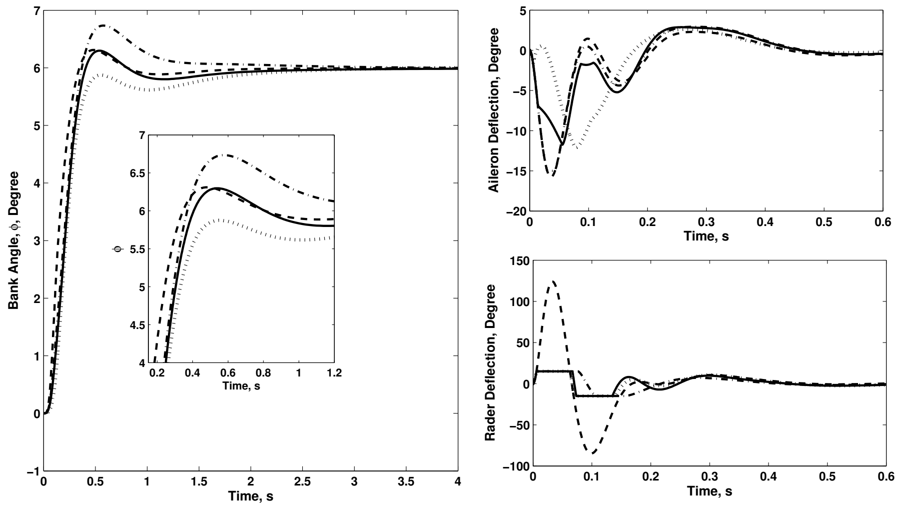

Fig. 7. System response (second example), output is in the left and inputs are in the right; unconstrained ideal closed-loop (dashed), constrained nominal closed-loop (dash-dotted), immediate activation of AW (dotted) and modified AW (solid)

IV. Conclusion

We proposed a new modified AW scheme, in which AW is applied when it is necessary and the performance of the saturated system faces substantial degradation. Here, we let the system stay saturated as long as the performance of the un-augmented system is expected to be adequate. Beyond this point, we apply a static AW to ensure the stability and performance of the saturated system. The design point, g_d, depends on how much input uncertainty the nominal closed-loop can tolerate. The results of simulation for two standard examples show that the new scheme can work better than the traditional anti-windup especially when actuator commands are only moderately higher than saturation bounds.

Appendix

Lemma 1 ([3])**.** *Given a square matrix D, if $-2I + D + D^T < 0$, then $I - D\Delta$ is nonsingular for all Δ such that $z \mapsto \Delta z$ belong to sector $[0, I]$.*

Consider $u = C_u x + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda)q$. We replace q by its value $q = \Delta u$:

$$u = C_u x + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda)\Delta u$$

where Δ is the deadzone nonlinearity. Moving the u term to one side, we obtain

$$(I - (D_{uq} - D_{u\eta}\Lambda)\Delta)u = C_u x + D_{uw} w \quad .$$

Thus, for well-posedness and uniqueness of the solution, we need $I-(D_{uq}-D_{u\eta}\Lambda)\Delta$ nonsingular. Then, the well-posedness condition for the static AW from Lemma 1 becomes $-2I + (D_{uq} - D_{u\eta}\Lambda) + (D_{uq} - D_{u\eta}\Lambda)^T < 0$ or in the weighted form

$$-2M + D_{uq}M - D_{u\eta}X + MD_{uq}^T - X^T D_{u\eta}^T < 0. \tag{21}$$

Note that (21) can be written as

$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} -M & MD_{uq}^T - X^T D_{u\eta}^T \\ D_{uq}M - D_{u\eta}X & -M \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} < 0.$$

Then, block (2:3,2:3) of (8) is the sufficient condition for well-posedness.

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