Modified Dynamic Anti-windup through Deferral of Activation

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SUMMARY

We consider the dynamic Anti-Windup design problem for linear systems with saturating actuators. The basic idea, proposed here, is to apply Anti-Windup only when the performance of saturated system faces substantial degradation. We provide synthesis LMIs to obtain the gains of the dynamic Anti-Windup compensator in a structure that delays the activation of the Anti-Windup. Benefits of the proposed design method over the immediate activation of the Anti-Windup are demonstrated using a well-known example.

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1. INTRODUCTION

Figure 1 shows the most common structure of Anti-Windup (AW) argumentation for stable linear plants with bounded actuators, where sat$(\cdot)$ represents the standard decentralized saturation nonlinearity. In this structure $\sum_c$ is the *nominal* controller, designed without any regard to the saturation bounds of the actuators. Therefore, this controller can be easily designed using linear control theory, and can provide desirable performance in small signal regime. However, as it is likely to saturate for some operating conditions, it is augmented with a so-called AW protection loop. Recently, using LMI characterization and effective numerical solvers, rigorous results on stability and performance measures, mostly $\mathcal{L}_2$ gain, have been achieved for the cases where the augmentation considered is static, or dynamic with an order matching that of the plant (e.g., [1]-[6]). Although the static augmentation complies more with the basic AW approach (no interference with the nominal controller when it is not saturated), as shown in [4], it is feasible if and only if there exists a quasi-common Lyapunov function between the open-loop system and the unconstrained (nominal) closed-loop system. As a result of this condition, there are problems in which designing AW with static gains is not feasible. In contrast, in case of the dynamic AW, the problem is always feasible if the open-loop plant is stable (see [4] and the numerical example below). Providing a

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modified scheme to improve the performance of this dynamic AW is one of the key objectives of this paper.

One of the main features of the AW scheme of Fig. 1 (we refer to it sometimes as the *traditional* AW) is that as soon as saturation occurs, the AW loop is activated, and usually replaces a high-gain (high performance) controller with a controller whose performance guarantee is no better than the open-loop system (see, [4]). However, sometimes nominal controllers are best left alone to deal with modest levels of saturation, since the AW interference can degrade the performance (e.g., see the numerical example of [5], or the first 5 seconds of the single input numerical example of [6]). The benefits of over-saturated controllers are exploited in e.g. [7], [8], and [9]. However, these results are in form of a priori approaches to the saturation problem, i.e. approaches which consider input saturation as a constraint in the very beginning of control design, often obtaining controllers that have substantially lower performance in the small signal region.

Recently, the idea of deferral of activation of the AW in the structure of Fig. 1 was presented in [10] for static AW case (see Fig. 2). In this modified structure, by introducing an artificial saturation element with larger limits, the activation of the AW has been postponed to a point that the assistance of AW is needed to preserve stability and appropriate performance. One can show the moderate levels of saturation can be modeled as mild structured uncertainty in the input matrices of the plant, i.e., in a 'matched' uncertainty form. Therefore, the level of the deferral depends on the robustness of the nominal controller.

The difference between structures of Fig. 1 and Fig. 2 can be viewed as the following: in the former, AW is activated as soon as saturation is encountered, resulting in a safe (stable) but typically low performance controller; while in the later, the nominal controller acts as a high performance controller subjected to a modest amount of parameter uncertainty at the input, and AW activates when the system goes beyond the reasonable tolerance of the system to the uncertainty.

In this note, we expand the results of [10] in several aspects: (i) we extend the result to the case of dynamic AW gains to address cases in which the static gain solution is not feasible, (ii) we provide a single LMI formulation of the problem, compared to the 2^{n_u} LMIs needed in [10], along with the resulting tradeoffs, and (iii) discuss the feasibility conditions as well as the order of the dynamic AW and compare these to the traditional AW approach.The results are evaluated through an illustrative example.

2. PROBLEM DEFINITION AND THE NEW STRUCTURE

Consider an open-loop stable plant with the following state space representation where $x_p \in R^{n_p}$ is the plant state vector, $\hat{u} \in R^{n_u}$ is the input vector, $w \in R^{n_w}$ is the exogenous input vector, and $y \in R^{n_y}$ and $z \in R^{n_z}$ are measured and controlled outputs, respectively:

$$\Sigma_p \quad \sim \quad \begin{cases} \dot{x}_p = A_p x_p + B_1 w + B_2 \hat{u} \\ y = C_2 x_p + D_{21} w \\ z = C_1 x_p + D_{11} w + D_{12} \hat{u} \end{cases} \tag{1}$$

We call the system with $\hat{u} \equiv 0$ the *open-loop* system. Throughout, we assume that a *nominal* linear controller has been designed for this system. In absence of saturation, the nominal controller guarantees the stability of the system and renders a highly desirable performance. However, since it is prone to saturation in some operating conditions, an AW block is needed. AW is introduced to the nominal controller by adding the correction terms η_1 and η_2 to the state and output equations, respectively:

$$\widehat{\Sigma}_c \sim \begin{cases} \dot{x}_c = A_c x_c + B_{cy} y + B_{cw} w + \eta_1 \\ u = C_c x_c + D_{cy} y + D_{cw} w + \eta_2 \end{cases} . \tag{2}$$

Here, $x_c \in R^{n_c}$ is the controller state vector and $u \in R^{n_u}$ is the output vector of the controller. Thus, nominal controller is (2) when $\eta_1 = \eta_2 = 0$. The saturation is assumed decentralized with the saturation limit u_{lim} for each u_i $(i = 1, 2, ..., n_u)$:

$$\hat{u}_i = sat(u_i) = sgn(u_i)\, min\{|u_i|, u_{lim}\}.$$

In the traditional AW (Fig. 1), the difference between the output of the controller and the input to the plant is used to activate the AW, resulting in the immediate AW interference. As discussed earlier, the immediate application of AW is not always necessary, and sometimes leads to the loss of performance. In [10] a modified structure of AW is introduced to postpone the activation of AW (see Fig. 2), along with the synthesis LMIs in the case of static AW. Unfortunately, in some problems, the search for the static gains is not feasible. To generalize the results, in this paper we consider the dynamic AW case, i.e. when AW is:

$$\Sigma_a \sim \begin{cases} \dot{x}_a = A_a x_a + B_a q \\ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = C_a x_a + D_a q \end{cases} \tag{3}$$

where $x_a \in R^{n_a}$ is the state vector, $q \in R^{n_u}$ is the input vector, and $\eta \in R^{n_c + n_u}$ is the output vector of the AW.

In the following, for clarity, we first consider the single input plants. Consider the modified AW structure of [10], shown in Fig. 2. By adding an artificial saturation element with bounds $\pm\frac{1}{g_d}u_{lim}$, where $0 < g_d < 1$ is the design point picked by designer, we delay the activation of the AW to beyond $\pm\frac{1}{g_d}u_{lim}$. The level of deferral, i.e. the value of g_d, depends on the performance robustness of the nominal controller. For details on how to choose the design point g_d, see [10]. The signal to activate the AW in the modified structure is $q = u - u_d$, where u_d is the output of the artificial saturation element.

The goal thus is to find the dynamic AW matrices, i.e., A_a, B_a, C_a, and D_a, which make the closed-loop system of Fig. 2 stable and also provide desirable $\mathcal{L}_2$ gain. Here, we use $\mathcal{L}_2$ gain as a measure of performance, as is common in many references. With relative ease, one can make modest alteration to use an estimate of peak-to-peak or energy-to-peak gains, as well.

Fig. 2 has two nonlinear elements, associated with the two saturations. We can model the real actuator by the time varying gain $G(t)$ ([9], [11]): since the signal entering it has a magnitude of

u_{lim}/g_d, we have

$$\hat{u}(t) = G(t)u_d(t), \quad G(t) = \begin{cases} 1 & |u_d(t)| \le u_{lim} \\ sgn(u(t))\frac{u_{lim}}{u(t)} & |u_d(t)| > u_{lim} \end{cases} \tag{4}$$

It is clear that when the actuator is not saturated, $G(t) = 1$. Since the maximum command $|u(t)|$ received by the real actuator is $\frac{1}{g_d}u_{lim}$, then the minimum value retained by $G(t)$ is g_d, i.e., $G(t) \in [\, g_d\,,\, 1]$. This allows us to replace the saturation box with a single gain varying between g_d and 1. This gain can be calculated on line by comparing the actuator commands with u_{lim} and u_{lim}/g_d. Therefore, the resulting model is a typical (quasi) linear parameter varying model. In the case of a system with multiple actuators, the saturation nonlinearity can be modeled as a diagonal matrix $G(t)$ where each diagonal element is a gain of the form (4) with lower bound g_{d_i} corresponding to the level of deferral chosen for that channel. Here, we make no assumptions regarding the disturbances or reference signals. As a result, the first, artificial, saturation can be considered as a time varying gain that varies between zero and one. This would add a second parameter in the model. To avoid using multiple parameters, which can lead to very large number of LMIs, and to be consistent with the structure of the traditional AW approach, we treat the saturation element whose error signal is used for anti-windup action (i.e., the artificial one here) as a sector nonlinearity only. Here, we consider the two cases where actual saturation element is modeled as either a polytopic nonlinearity or as a sector nonlinearity.

Let us consider the case in which the actual saturation is modeled as polytopic uncertainty while the artificial saturation is modeled as a bounded dead-zone nonlinearity. Defining the overall state space vector $\tilde{x} = [x_p^T \;\; x_c^T \;\; x_a^T]^T$, we can represent the closed-loop system $\Sigma(G)$ as:

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}(G(t))\tilde{x} + \tilde{B}_w(G(t))w + (\tilde{B}_q(G(t)) - \tilde{B}_\eta(G(t))\Lambda)q \\ z = \tilde{C}_z(G(t))\tilde{x} + \tilde{D}_{zw}(G(t))w + (\tilde{D}_{zq}(G(t)) - \tilde{D}_{z\eta}(G(t))\Lambda)q \\ u = \tilde{C}_u\tilde{x} + \tilde{D}_{uw}w + (\tilde{D}_{uq} - \tilde{D}_{u\eta}\Lambda)q \\ q = \Delta\, u \end{cases} \tag{5}$$

where $\Lambda = [B_a^T \;\; D_a^T]^T$. See Appendix I for the system matrices of (5). Note that matrices $\tilde{A}(G(t))$, $\tilde{C}_z(G(t))$, and $\tilde{C}_u$ contain the AW matrices A_a and/or C_a.

Alternatively, the actual saturation element could be modeled as sector nonlinearity with Sector $[G_d, I]$ where $G_d = diag[g_{d_1}, \cdots, g_{d_{nu}}]$. In the following, we use a standard transformation to turn this nonlinearity into a symmetric Sector $[-I, I]$. After some manipulations we get

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}_h\tilde{x} + \tilde{B}_{hw}w + \tilde{B}_{hp}p + (\tilde{B}_{hq} - \tilde{B}_{h\eta}\Lambda)q \\ z = \tilde{C}_{hz}\tilde{x} + \tilde{D}_{hzw}w + \tilde{D}_{hzp}p + (\tilde{D}_{hzq} - \tilde{D}_{hz\eta}\Lambda)q \\ u = \tilde{C}_u\tilde{x} + \tilde{D}_{uw}w + (\tilde{D}_{uq} - \tilde{D}_{u\eta}\Lambda)q \\ u_d = (u - q) = \tilde{C}_u\tilde{x} + \tilde{D}_{uw}w + (\tilde{D}_{uq} - I - \tilde{D}_{u\eta}\Lambda)q \\ q = \Delta u \\ p = H(t)u_d \end{cases} \tag{6}$$

where

$$H(t) = 2(I - G_d)^{-1}(G(t) - \frac{1}{2}(I + G_d)) \tag{7}$$

is used to obtain a Sector $[-I, I]$ nonlinearity. System matrices used in (6) are listed in Appendix II. There, we have defined $H_d = \frac{1}{2}(I + G_d)$, where the index 'h' is to show the dependency of the corresponding system matrix to variable H_d (or G_d). In the next section, we will obtain synthesis inequalities for the two models in (5) and (6) and discuss the relative benefits.

In the following subsections, we try to motivate considering saturation as an uncertainty. For clarity, we assume single actuator when explaining the techniques. However, the results are applicable for the multi-actuator cases as well (by replacing g_d with diagonal G_d).

2.1. Saturation as matched uncertainty: Polytpoic case

The principle idea is that the nominal closed-loop system typically has good robustness margins, particularly with respect to 'match uncertainty' at the input (i.e., when uncertain terms have B_2 factors). Consider the two models discussed above (i.e., (5) and (6)) where there is no AW, or equivalently, when $q = 0$. In that case, the saturation acts as a single match uncertainty modeled either as a sector type nonlinearity or polytopic one. Suppose *for some reason* – for example by bounds on the reference/disturbance – it could be shown that the command is only modestly larger than the saturation bound. In that case the saturation element acts as a small multiplicative (matched) uncertainty at the input and thus would have modest impact on stability and performance.

Consider the *nominal* system, with the original saturation element. We use (5) but remove the AW dynamics and artificial saturation element (so $x = [x_p^T \quad x_c^T]^T$ and q is set to zero in (5)). As a result, we get

$$\begin{cases} \dot{x} = A(G(t))x + B_w(G(t))w \\ z = C_z(G(t))x + D_{zw}(G(t))w \end{cases} \tag{8}$$

where matrix $A(G(t))$, etc are the nominal closed loop matrices containing the time varying uncertainty $G(t)$ – see Appendix I for details.

To study the performance of the saturated nominal closed-loop system for different levels of saturation, under the assumption that $|u(t)| \leq \frac{1}{g}u_{lim}$ for all t, we can use the standard bounded real inequality in which an estimate for the $\mathcal{L}_2$ gain of this closed-loop can be obtained from minimizing γ_g subject to $\mathbf{Q} > 0$ and

$$\begin{pmatrix} \mathbf{Q}A(G(t))^T + A(G(t))\mathbf{Q} & \star & \star \\ B_w(G(t))^T & -\gamma_g I & \star \\ C_z(G(t))\mathbf{Q} & D_{zw}(G(t)) & -\gamma_g I \end{pmatrix} < 0 \tag{9}$$

Due to linearity, the inequality above needs to be satisfied at the corners of the hypercube; i.e., a sufficient and necessary condition for (9) to hold for all $g_d \leq G(t) \leq 1$, is for it to hold for two values of $G(t) = g_d$ and $G(t) = 1$. If there are multiple inputs, the size of problem becomes prohibitively large (2^{n_u} matrix inequalities). Partly to address this issue, we will discuss using a sector bounded approach in the next subsection.

Fig. 3 shows the performance of the nominal constrained closed-loop system (without AW assistance) for the numerical example used later in this paper (see Section 4). In this figure, each point (g_d, γ) on the curve represents the performance of the system for $G(t) \in [\,g_d\,,\,1]$, i.e. if the actuator is guaranteed to receive – *somehow* – control command with peak value of $\frac{1}{g_d} u_{lim}$ or less. Figure 3 suggests that such a constrained nominal closed-loop system has adequate performance up to about $g_d = 0.8$, or equivalently $|u(t)| \leq \frac{u_{lim}}{0.8}$ (i.e., $G(t) \in [0.8\,,\,1]$), without any assistance from any external compensator. Of course, in general and without further modification, we cannot know or limit the peak value of the command to the actuator.

2.2. *Saturation as sector bounded uncertainty*

Naturally, the inequality (9) above can be written to correspond to a Sector $[g_d, 1]$. In order to be consistent with the results of [10] and the rest of this paper, we will follow the same centering and scaling used in (6). The goal is to calculate the $\mathcal{L}_2$ gain of the nominal constrained closed-loop system for different values of g_d, i.e., $g_d \leq G(t) \leq 1$ (for multi-actuator system we use the matrix G_d). Consider the *nominal* constrained closed-loop system, i.e, Fig. 1 when there is no AW protection in the loop. We replace saturation with its equivalent gain $G(t)$ described above, and then use the same $H(t)$ as defined in (7) to obtain a $[-1, 1]$ sector. With closed-loop state defined as $x = [x_p^T \quad x_c^T]^T$, the result will be a simplified form of state equation (6) in which $q = 0$ and the states associated with the AW dynamics are eliminate; i.e.,

$$\begin{cases} \dot{x} = A_h x + B_{hw} w + B_{hp} p \\ z = C_{hz} x + D_{hzw} w + D_{hzp} p \\ u = C_u x + D_{uw} w + D_{up} p \\ p = H(t) u \end{cases} \tag{10}$$

Again, please see Appendix II for the system matrices. Note that $H(t)$ is a diagonal matrix of dimension n_u with each diagonal elements taking values between -1 and 1, therefore, $H(t) \in [-1, 1]_W$; i.e,

$$(p + u)^T W (p - u) \leq 0$$

where W is positive scalar (in case of multi-input systems, positive definite diagonal) scale. Using a quadratic Lyapunov function $V = x^T Q^{-1} x$ and following standard results along with invoking the S-procedure, for a positive τ, the $\mathcal{L}_2$ gain of the nominal constrained closed-loop system from w to z can be obtained as:

$$\frac{d}{dt} x^T Q^{-1} x + \gamma^{-1} z^T z - \gamma w^T w - \tau (p + u)^T W (p - u) < 0$$

After expanding this inequality in terms of the constrained closed-loop system matrices, and using $M = \tau^{-1} W^{-1}$, it can be represented in the LMI form:

$$\begin{pmatrix} A_h \mathbf{Q} + \mathbf{Q} A_h^T & B_{hw} & \mathbf{Q} C_{hz}^T & B_{hp} \mathbf{M} & \mathbf{Q} C_u^T \\ \star & -\gamma I & D_{hzw}^T & 0 & D_{uw}^T \\ \star & \star & -\gamma I & D_{hzp} \mathbf{M} & 0 \\ \star & \star & \star & -\mathbf{M} & 0 \\ \star & \star & \star & \star & -\mathbf{M} \end{pmatrix} < 0. \tag{11}$$

For the numerical example of Section 4, (11) yields a constrained nominal closed-loop performance plot very similar – essentially identical – to that of Fig. 3. As discussed below however, in the synthesis problems the two approaches (i.e., modeling actual saturation as polytopic vs. sector bounded elements) yield significantly different results.

2.3. Remarks

In cases where we do not have bounds on the references and/or disturbances, we cannot assume a known value for g_d. The artificial saturation box in Fig. 2 is added precisely for this purpose since by using a saturation limit of $\pm\frac{1}{g_d}u_{lim}$, something that is easily done with software, we can ensure that the actual saturation element does not see a signal magnitude larger than $\frac{1}{g_d}u_{lim}$. Of course, we need to incorporate this new saturation nonlinearity in the synthesis. If we model this artificial saturation as another polytopic uncertainty, it will result in 4 inequalities of the form in (9), for each actuator! To avoid such problems, we add the new saturation simply as a deadzone nonlinearity in Sector $[0, 1]$, both to the polytopic form in (9) and to the sector nonlinearity form in (11).

It is important to keep in mind that the equations in the above two subsections were meant as the motivation for the general approach used here. The synthesis results of Section 3 are based on the model and notation used in (5) and (6).

3. LMI-BASED SYNTHESIS

3.1. Synthesis for the polytpoic case

For the system of (5), we use $V = \tilde{x}^T Q^{-1}\tilde{x}$ as the Lyapunov function and use the standards approach to establish an upper bound for the $\mathcal{L}_2$ gain:

$$\frac{d}{dt}(\tilde{x}^T Q^{-1}\tilde{x}) + \gamma^{-1}z^T z - \gamma w^T w + 2\tau q^T \tilde{W}(u-q) < 0 \tag{12}$$

where the last term on the left hand side denotes the use of S-procedure to incorporate the sector bounded uncertainty associated with the artificial saturation. Following routine manipulations (see [13] for details) we obtain the following matrix inequality as a sufficient condition for (12)

$$\begin{pmatrix} \tilde{\mathbf{A}}(G(t))\mathbf{Q} + \mathbf{Q}\tilde{\mathbf{A}}(G(t))^T & \star & \star & \star \\ \tilde{B}_w(G(t))^T & -\gamma I & \star & \star \\ \tilde{\mathbf{C}}_z(G(t))\mathbf{Q} & \tilde{D}_{zw}(G(t)) & -\gamma I & \star \\ \Phi_{41}(G(t)) & \tilde{D}_{uw} & \Phi_{43}(G(t)) & \Phi_{44}(G(t)) \end{pmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \Phi_{41}(G(t)) &= \tilde{\mathbf{M}}\tilde{B}_q(g_d)^T - \tilde{\mathbf{M}}\mathbf{\Lambda}^T\tilde{B}_\eta(G(t))^T + \tilde{\mathbf{C}}_u\mathbf{Q} \\ \Phi_{43}(G(t)) &= \tilde{\mathbf{M}}\tilde{D}_{zq}(G(t))^T - \tilde{\mathbf{M}}\mathbf{\Lambda}^T\tilde{D}_{z\eta}(G(t))^T \\ \Phi_{44}(G(t)) &= -2\tilde{\mathbf{M}} + \tilde{D}_{uq}\tilde{\mathbf{M}} + \tilde{\mathbf{M}}\tilde{D}_{uq}(G(t))^T - \tilde{D}_{u\eta}\mathbf{\Lambda}\tilde{\mathbf{M}} - \tilde{\mathbf{M}}\mathbf{\Lambda}^T\tilde{D}_{u\eta}^T \end{aligned} .$$

where $1 \geq G(t) \geq g_d$, and $\tilde{\mathbf{M}} = \tau^{-1}\tilde{\mathbf{W}}^{-1}$.

To transform this matrix inequality into a convex search for the dynamic AW gains, we assume that the dynamic AW is of full order, i.e., $n_a = n_p + n_c$. Without loss of generality, we use a series of, by now, routine transformations and manipulations including using the following structure for Q:

$$Q = \begin{bmatrix} Y & S \\ S & S \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} Z & -Z \\ -Z & Z + S^{-1} \end{bmatrix} \tag{14}$$

where S, Y, Z are symmetric positive definite square matrices of order $(n_p + n_c)$ and $S = Y - Z^{-1}$. After some manipulations (omitted for brevity), we obtain the following for the polytopic model for single actuator case.

Theorem 1 (Synthesis condition for polytopic model)
Consider the following matrix inequality with matrix variables $\mathbf{F}_i, i = 1, \ldots 4$, of appropriate dimensions along with $\mathbf{Y}$, and $\mathbf{S}$ positive definite and $\tilde{\mathbf{M}}$ and γ positive scalars that satisfy:

$$\begin{pmatrix} \Omega_{11}(.) & \star & \star & \star & \star \\ \Omega_{12}(.)^T & \mathbf{F}_1 + \mathbf{F}_1^T & \star & \star & \star \\ B_w(.)^T & 0 & -\gamma I & \star & \star \\ \Omega_{41}(.) & \Omega_{42}(.) & D_{zw}(.) & -\gamma I & \star \\ \Omega_{51}(.) & \Omega_{52} & D_{uw} & \Omega_{54}(.) & \Omega_{55} \end{pmatrix} < 0 \tag{15}$$

where

$$\begin{aligned}
\Omega_{11}(.) &= A(.)\mathbf{Y} + \mathbf{Y}A(.)^T + B_\eta(.)\mathbf{F}_2 + \mathbf{F}_2^T B_\eta(.)^T \\
\Omega_{12}(.) &= A(.)\mathbf{S} + B_\eta(.)\mathbf{F}_2 + \mathbf{F}_1^T \\
\Omega_{41}(.) &= C_z(.)\mathbf{Y} + D_{z\eta}(.)\mathbf{F}_2 \\
\Omega_{42}(.) &= C_z(.)\mathbf{S} + D_{z\eta}(.)\mathbf{F}_2 \\
\Omega_{51}(.) &= \tilde{\mathbf{M}}B_q(.)^T + \mathbf{F}_4^T B_\eta(.)^T + C_u\mathbf{Y} + D_{u\eta}\mathbf{F}_2 \\
\Omega_{52} &= \mathbf{F}_3^T + C_u\mathbf{S} + D_{u\eta}\mathbf{F}_2 \\
\Omega_{54}(.) &= \tilde{\mathbf{M}}D_{zq}(.)^T + \mathbf{F}_4^T D_{z\eta}(.)^T \\
\Omega_{55} &= -2\tilde{\mathbf{M}} + D_{uq}\tilde{\mathbf{M}} + \tilde{\mathbf{M}}D_{uq}^T + D_{u\eta}\mathbf{F}_4 + \mathbf{F}_4^T D_{u\eta}^T
\end{aligned}.$$

with matrices $A(.)$, $B_\eta(.)$, denote $A(G(t))$, etc for $1 \geq G(t) \geq g_d$. (see Appendix I for details). Then, the following anti-windup gains guarantee an $\mathcal{L}_2$ gain of γ from w to z:

$$\mathbf{A}_a = \mathbf{F}_1\mathbf{S}^{-1}, \;\; \mathbf{C}_a = \mathbf{F}_2\mathbf{S}^{-1}, \;\; \mathbf{B}_a = \mathbf{F}_3\tilde{\mathbf{M}}^{-1}, \;\; \mathbf{D}_a = \mathbf{F}_4\tilde{\mathbf{M}}^{-1}. \tag{16}$$

3.2. Synthesis for the sector nonlinearity case

For the sector bounded model in (6), we follow the general approach used in the preceding sections. For a quadratic Lyapunov function $V = \tilde{x}^T Q^{-1}\tilde{x}$, the $\mathcal{L}_2$ gain from w to z for the closed-loop system of Fig. 2 with dynamic AW can be obtained by minimizing γ with respect to:

$$\frac{d}{dt}\tilde{x}^T Q^{-1}\tilde{x} + \gamma^{-1}z^T z - \gamma w^T w - \tau_1(p - u_d)^T W(p + u_d) - 2\tau_2 q^T \tilde{W}(q - u) < 0 \tag{17}$$

where, similar to (12), the last term on the left hand side denotes the use of S-procedure to incorporate the sector bounded uncertainty associated with the artificial saturation, and the term before it denotes the use of S-procedure to incorporate the sector bounded uncertainty associated with the actual saturation. Expanding this inequality in terms of the closed-loop system matrices, this condition can be represented in the following equivalent matrix inequality form (after some routine manipulations)

$$\begin{pmatrix} \tilde{\mathbf{A}}_h\mathbf{Q}+\mathbf{Q}\tilde{\mathbf{A}}_h^T & \tilde{B}_{hw} & \mathbf{Q}\tilde{\mathbf{C}}_{hz}^T & \tilde{B}_{hp}\mathbf{M} & \mathbf{Q}\tilde{\mathbf{C}}_u^T+\tilde{B}_{hq}\tilde{\mathbf{M}}-\tilde{B}_{h\eta}\mathbf{X} & \mathbf{Q}\tilde{\mathbf{C}}_u^T \\ \star & -\gamma I & \tilde{D}_{hzw}^T & 0 & \tilde{D}_{uw}^T & \tilde{D}_{uw}^T \\ \star & \star & -\gamma I & \tilde{D}_{hzp}\mathbf{M} & \tilde{D}_{hzq}\tilde{\mathbf{M}}-\tilde{D}_{hz\eta}\mathbf{X} & 0 \\ \star & \star & \star & -\mathbf{M} & 0 & 0 \\ \star & \star & \star & \star & \begin{matrix} -2\tilde{\mathbf{M}}+\tilde{D}_{uq}\tilde{\mathbf{M}}-\tilde{D}_{u\eta}\mathbf{X} \\ +\tilde{\mathbf{M}}\tilde{D}_{uq}^T-\mathbf{X}^T\tilde{D}_{u\eta}^T \end{matrix} & \tilde{\mathbf{M}}D_{uq}^T-\tilde{\mathbf{M}}-\mathbf{X}^T\tilde{D}_{u\eta}^T \\ \star & \star & \star & \star & \star & -\mathbf{M} \end{pmatrix} < 0 \tag{18}$$

where $\mathbf{X} = \begin{bmatrix} \mathbf{B}_a\tilde{\mathbf{M}} \\ \mathbf{D}_a\tilde{\mathbf{M}} \end{bmatrix}$ while $\tilde{\mathbf{M}} = \tau_2^{-1}\tilde{\mathbf{W}}^{-1}$ and $\mathbf{M} = \tau_1^{-1}\mathbf{W}^{-1}$ are scales as before. Note that $\tilde{\mathbf{A}}_h$, etc. contain A_a and C_a (see Appendix II for details), thus the form above is not linear in the variables. However, we can use a procedure quite similar to the one used in the previous subsection (i.e. same structure as in (14)) to convexify (18). The result will be a single LMI whose solution leads to an AW order of $n_p + n_c$. Due to the similarity with Theorem 1, such routine manipulations are omitted. In the following subsection, we present an alternative manipulation of (18) which provides interesting insights and connections to the traditional AW scheme.

3.3. Remarks

The approach of the previous subsection (using sector nonlinearity) contains a single matrix inequality, while Theorem 1 needs to be satisfied for all values of $G(t)$. Due to linearity, this requires satisfaction only at the extreme values of g_d and 1, i.e., two inequalities of the size in (15), in the single input case. In the multi-input case, this can lead to potentially large number of LMI's (i.e., 2^{n_u}), albeit with somewhat smaller dimensions – see [13] for further reduction in dimension. Also note that due to the structure used in (14), both techniques lead to AW dynamics of order $n_p + n_c$, though since the gain matrices appear explicitly, they are consistent with use in multi-objective problems (where other objectives and corresponding constraints are simply added).

As discussed in Section 3.1, the main sufficient condition in polytopic uncertainty form is the satisfaction of (13), for all values of $G(t)$. Now this can be rewritten as

$$\Psi(G(t)) + \mathcal{G}^T\,\bar{\Lambda}^T\,\mathcal{H}(G(t)) + \mathcal{H}(G(t))^T\,\bar{\Lambda}\,\mathcal{G} < 0$$

where $\bar{\Lambda} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix}$. To eliminate the AW gain matrices (i.e., A_a, etc.) we can apply the projection lemma to obtain a set of conditions very similar to those in [4], except one of the inequalities would have dependency on $G(t)$. Satisfaction of these conditions, however, do not mean existence of an LTI AW compensator since the set of all solutions for $\bar{\Lambda}$ will be of the form:

$$\bar{\Lambda} = -\Pi^{-1}\mathcal{H}(G(t))\Phi(G(t))\mathcal{G}^T\Theta + \Omega(G(t))^{-1/2}L\Theta(G(t))^{1/2} \tag{19}$$

where

$$\Phi(G(t)) = (\mathcal{H}(G(t))\Pi^{-1}\mathcal{H}(G(t))^T - \Psi(G(t)))^{-1} > 0, \quad \Pi > 0, \quad \|L\| < 1$$

$$\Omega(G(t)) = \Pi^{-1} - \Pi^{-1}\mathcal{H}(G(t))(\Phi(G(t)) - \Phi(G(t))\mathcal{G}^T\Theta(G(t))\mathcal{G}\Phi(G(t)))\mathcal{H}(G(t))\Pi^{-1}$$

$$\Theta(G(t)) = (\mathcal{G}\Phi(G(t))\mathcal{G}^T)^{-1}$$

The dependence of $\bar{\Lambda}$ above on $G(t)$ results in a situation similar to that encountered in robust dynamic output feedback problem: one has to enforce conditions on $L \neq 0$ (or second term in (19)) so that the *same* constant $\bar{\Lambda}$ holds for all $G(t)$. This of course destroys the convexity of the problem. Note that similar to the LPV problem, if one was willing to have $\bar{\Lambda}$ be parameter-varying (i.e., explicit dependence on $G(t)$), then this approach could be used to obtain AW with $n_a = n_p$ but the resulting LPV form of the AW compensator will have a far more complicated controller structure and implementation (and would not be extendable to the multi-objective problems, either). While this might be of some interest, clearly it is beyond the scope of this note: designing a relatively simple LTI Anti-windup compensation.

However, applying standard projection lemma to the inequality in (18), it is relatively straightforward (following the steps in [4]) to transform it into smaller LMIs in which the AW gains do not appear explicitly. There, this allowed the study of certain properties such as the need for stable open loop (as the sufficient condition for existence of a solution). More importantly, it was shown that there are solutions for anti-windup orders of n_p or larger. As we discussed earlier, to facilitate extension to multi-objective problems, we used the change of variable approach (from [12]), which automatically results in a dynamic AW of order $n_a = n_p + n_c$. As discussed above, the projection lemma is not applicable to the polytopic form, and the result guarantees only order $n_a = n_p + n_c$. However, as the example below shows, using sector nonlinearity can result is significant conservatism – which is not surprising since polytopic models are less conservative that those from sector conditions (see for example [14]).

Following the steps used in [4] and applying them to (18), we obtain the following.

Theorem 2 (Existence condition for the sector nonlinear case)
Given the system in (6), an integer $n_a \geq 0$ scalars $0 \leq g_{d_i} < 1$ $(i = 1, ..., n_u)$, and positive scalar $\tilde{\gamma}$, there exists a linear anti-windup compensator of order n_a that guarantees well-posedness and quadratic performance level of $\tilde{\gamma}$ if and only if there exist a positive definite matrix S and $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$ where $R - S \geq 0$ which the following LMI set is feasible for them:

$$(11) \text{ with } \mathbf{Q} \text{ replaced with } \mathbf{S} \tag{20}$$

$$rank[\mathbf{R} - \mathbf{S}] \leq n_a \tag{21}$$

$$\begin{pmatrix} A_p\mathbf{R}_{11} + \mathbf{R}_{11}A_p^T & B_1 & \mathbf{R}_{11}C_1^T & B_2(I - H_d)\mathbf{M} \\ \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & -\gamma I & D_{12}(I - H_d)\mathbf{M} \\ \star & \star & \star & -\mathbf{M} \end{pmatrix} < 0 \tag{22}$$

$$\gamma \leq \tilde{\gamma} \tag{23}$$

Proof

The proof is through elimination the dynamic AW matrices via preforming elimination Lemma on (18). The details are omitted for brevity, since it follows closely the steps in [4]. □

Except for the rank condition the problem above is linear with respect to the unknowns. For $n_a \geq n_p$, by exploiting the special structure of the AW design problem, one can show that the rank condition can be replaced with equivalent linear conditions (see proposition 2 of [4]). This will allow us to obtain smaller order AW using an efficient LMI solver.

Note that for feasibility in the traditional approach ([4]) only open loop stability was needed, since assuming closed loop nominal stability is rather trivial. Given the basic idea of allowing saturation to act as a mild uncertainty, it is not surprising that we get nominal open loop and closed loop *robust* stability as sufficient and necessary condition. Recall that (20) is the sufficient condition for nominal closed loop stability in the presence of the uncertainty due to the saturation, while it is easy to see that (22) is simply the sufficient condition for the nominal open loop robust stability (given uncertainty due to $G(t)$). As with the traditional approach, dynamic structure of AW leads to different Lyapunov matrices for the two conditions (ie Q and R_{11}) thus avoiding the coupling condition faced in static AW gain structure. Unfortunately, in this case, we get a coupling due to M, i.e. the two conditions should be satisfied with the same M. While M does not alter convexity, it can lead to significant conservatism, as shown in the example below. In our experience, the extra efforts needed in the polytopic approach (more LMIs, higher order AW dynamics) are easily justified both in terms of guaranteed performance (i.e., γ) and actual performance (as verified by simulation). It is possible, however, that if there are numerous inputs or the order of the controller is quite large, the sector model can become more attractive.

Finally, note that in $\lim G(t)_{g_d \to 1}$, the traditional AW case is recovered, which is feasible if the open loop is stable. Since the results are based on strict nonlinearities, due to continuity, there is always some $0 < g_d < 1$ for which the proposed approach will work for open loop stable system. Generally, it is easy to obtain the ultimate g_d through a scaler line search.

Figure 4 compares the results of performance guarantees of the system with modified dynamic AW for different values of the design point g_d for the two proposed approaches, for the the example of Section 4. As this figure shows, the sector approach is very conservative compared to the LPV, when the synthesis results are considered. Neither case is feasible much beyond $g_d = 0.77$.

4. EXAMPLE

The following example is taken from [4]. Plant is defined as

$$\left[\begin{array}{c|c|c} A_p & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cccc|c|c} 0 & 1 & 0 & 0 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 & 0 & 2.71762 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -812.61 & -29.87 & -30.10 & 0 & 15.61 & 6.68268 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

Nominal controller is $A_c = A_p - B_2K - LC_2,\ \ B_{cy} = L,\ \ C_c = -K$ where

$$K = \begin{bmatrix} 64.81 & 213.12 & 1242.27 & 85.82 \end{bmatrix},\ \ L = \begin{bmatrix} 64 & 2054 & -8 & -1432 \\ -8 & -280 & 142 & 10169 \end{bmatrix}^T$$

and B_{cw}, D_{cy} and D_{cw} are zero matrices of appropriate dimension. The saturation bound is $u_{lim} = 5$, and given Fig. 3 and Fig. 4, we use $g_d = 0.77$.

For this example, static AW compensation is not feasible. However, in [4] a plant order dynamic AW is obtained which guarantees a performance level of $\gamma = 181.82$ (the numerical values of the AW matrices are given in [4]). The results of simulation for a step input of duration 0.1 and magnitude 0.21 (small input), and an step input of duration 0.1 and magnitude 0.8 (large input) are depicted in Fig. 5 (a) and (b), respectively. For small reference input, the constrained nominal closed-loop system (i.e., no AW in dotted line) behaves much better than the closed-loop system with *traditional* AW assistance. This indicates that for small input, thus lower levels of saturation absent any augmentation, the immediate interference of the *traditional* AW does not help and degrades the system response. However, for large input, the nominal constrained (no AW) degrades significantly and AW is needed.

The two cases show the main motivation of the proposed approach: delaying the start of the AW action when the natural robustness of the nominal controller can deal with the problem (relatively small reference signal) while providing a measure of assurance that AW can give for large reference signals. Comparing the response of the system with the traditional dynamic AW (dash-dotted line) against the response of the system with modified dynamic AW (solid line) in Fig. 5, shows that the delayed AW, by exploiting the input robustness of the nominal system, can produce a better performance, especially for small input case, as expected.

5. APPENDIX I

The system matrices in (5) and (9) are as follows:

$$\left[\begin{array}{c|c} \tilde{A}(G(t)) & \tilde{B}_w(G(t)) \\ \hline \tilde{C}_z(G(t)) & \tilde{D}_{zw}(G(t)) \\ \hline \tilde{C}_u & \tilde{D}_{uw} \end{array}\right] = \left[\begin{array}{cc|c} A(G(t)) & B_\eta(G(t))C_a & B_w(G(t)) \\ 0 & A_a & 0 \\ \hline C_z(G(t)) & D_{z\eta}(G(t))C_a & D_{zw}(G(t)) \\ \hline C_u & D_{u\eta}C_a & D_{uw} \end{array}\right]$$

$$\left[\begin{array}{c|c} \tilde{B}_q(G(t)) & \tilde{B}_\eta(G(t)) \\ \hline \tilde{D}_{zq}(G(t)) & \tilde{D}_{z\eta}(G(t)) \\ \hline \tilde{D}_{uq} & \tilde{D}_{u\eta} \end{array}\right] = \left[\begin{array}{c|c} B_q(G(t)) & -B_\eta(G(t))[0_{(n_u+n_c)\times n_a}\ \ I_{(n_c+n_u)}] \\ \hline D_{zq}(G(t)) & -D_{z\eta}(G(t))[0_{(n_u+n_c)\times n_a}\ \ I_{(n_c+n_u)}] \\ \hline D_{uq} & -D_{u\eta}[0_{(n_u+n_c)\times n_a}\ \ I_{(n_c+n_u)}] \end{array}\right]$$

where I_n is the identity matrix of dimension n and

$$\left[\begin{array}{c|c} A(G(t)) & B_w(G(t)) \\ \hline C_z(G(t)) & D_{zw}(G(t)) \\ \hline C_u & D_{uw} \end{array}\right] = \left[\begin{array}{cc|c} A_p + B_2G(t)D_{cy}C_2 & B_2G(t)C_c & B_1 + B_2G(t)D_{cw} + B_2G(t)D_{cy}D_{21} \\ B_{cy}C_2 & A_c & B_{cw} + B_{cy}D_{21} \\ \hline C_1 + D_{12}D_{cy}C_2 & D_{12}G(t)C_c & D_{11} + D_{12}G(t)D_{cy}D_{21} + D_{12}G(t)D_{cw} \\ \hline D_{cy}C_2 & C_c & D_{cy}D_{21} + D_{cw} \end{array}\right]$$

$$\left[\begin{array}{c|c} B_q(G(t)) & B_\eta(G(t)) \\ \hline D_{zq}(G(t)) & D_{z\eta}(G(t)) \\ \hline D_{uq} & D_{u\eta} \end{array}\right] = \left[\begin{array}{c|c} -B_2 G(t) & B_2 G(t)[0_{n_u \times n_c} \;\; I_{n_u}] \\ 0 & [I_{n_c} \;\; 0_{n_c \times n_u}] \\ \hline -D_{12} G(t) & D_{12} G(t)[0_{n_u \times n_c} \;\; I_{n_u}] \\ \hline 0 & [0_{n_u \times n_c} \;\; I_{n_u}] \end{array}\right]$$

6. APPENDIX II

The system matrices in (6) and (10) are as follows:

$$\left[\begin{array}{c} \tilde{A}_h \\ \hline \tilde{C}_{hz} \\ \hline \tilde{C}_u \end{array}\right] = \left[\begin{array}{cc} A_h & B_{h\eta} C_a \\ 0 & A_a \\ \hline C_{hz} & D_{u\eta} C_a \\ \hline C_u & D_{u\eta} C_a \end{array}\right], \quad \left[\begin{array}{c|c} \tilde{B}_{hw} & \tilde{B}_{hp} \\ \hline \tilde{D}_{hzw} & \tilde{D}_{hzp} \\ \hline \tilde{D}_{uw} & \tilde{D}_{up} \end{array}\right] = \left[\begin{array}{c|c} B_{hw} & B_{hp} \\ 0 & 0 \\ \hline D_{hzw} & D_{hzp} \\ \hline D_{uw} & D_{up} \end{array}\right]$$

$$\left[\begin{array}{c|c} \tilde{B}_{hq} & \tilde{B}_{h\eta} \\ \hline \tilde{D}_{hzq} & \tilde{B}_{hz\eta} \\ \hline \tilde{D}_{uq} & \tilde{D}_{u\eta} \end{array}\right] = \left[\begin{array}{c|c} B_{hq} & B_{h\eta}[0 \;\; I_{(n_c+n_u)}] \\ 0 & [I_{n_a} \;\; 0] \\ \hline D_{hzq} & D_{hz\eta}[0 \;\; I_{(n_c+n_u)}] \\ \hline 0 & D_{u\eta}[0 \;\; I_{(n_c+n_u)}] \end{array}\right].$$

where

$$\left[\begin{array}{c} A_h \\ \hline C_{hz} \\ \hline C_u \end{array}\right] = \left[\begin{array}{cc} A_p + B_2 H_d D_{cy} C_2 & B_2 H_d C_c \\ B_{cy} C_2 & A_c \\ \hline C_1 + D_{12} H_d D_{cy} C_2 & D_{12} H_d C_c \\ \hline D_{cy} C_2 & C_c \end{array}\right]$$

$$\left[\begin{array}{c|c} B_{hw} & B_{hp} \\ \hline D_{hzw} & D_{hzp} \\ \hline D_{uw} & D_{up} \end{array}\right] = \left[\begin{array}{c|c} B_1 + B_2 H_d D_{uw} & B_2(I - H_d) \\ B_{cw} + B_{cy} D_{21} & 0 \\ \hline D_{11} + D_{12} H_d D_{kuw} & D_{12}(I - H_d) \\ \hline D_{cw} + D_{cy} D_{21} & 0 \end{array}\right].$$

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