On Bounded Real Matrix Inequality Dilation

Solmaz Sajjadi-Kia and Faryar Jabbari

Abstract

We discuss a variation of dilated matrix inequalities for the conventional Bounded Real matrix inequality, and other similarly structured inequalities. Here, system matrices are separated from Lyapunov matrix to allow the use of different Lyapunov matrices in multi-objective and robust problems. The search involves a bounded scalar parameter that enters the problem nonlinearly, and is dealt with a line search. To demonstrate the benefits of the new dilated matrix inequalities over the conventional ones, an example of controller synthesis with $\mathcal{L}_2$-gain performance measure ($\mathcal{H}_\infty$ control) for a system with polytopic uncertainty (robust problem) has been studied. It is shown that for the resulting robust problem the performance obtained via the dilated form is at least equal to those of the conventional one. Also, the connection between the proposed dilated form and the Full Block S-procedure is discussed.

1 Introduction

Many control techniques such as $\mathcal{H}_\infty$ and $\mathcal{H}_2$ use a quadratic Lyapunov function ($V = x^T Y x \mid Y > 0$) to obtain (Linear) Matrix Inequality ((L)MI) characterizations (e.g., see [2]). The resulting MIs end up with non-convex entries that have products of the Lyapunov variables and system matrices including control variables. In order to keep the problem convex or similar to a convex search, the non-convex product term is usually replaced by a new variable. This causes some degree of conservatism in multi-objective and robust analysis/synthesis problems by forcing common Lyapunov matrices for all objectives (e.g. see [3, 8, 15]). To reduce this conservatism, recently, several researchers have developed dilated matrix inequalities where by introducing slack variable(s), the system and the Lyapunov variables are separated. Here, we denote the slack variable G, the Lyapunov variable in a dilated MI $\mathcal{X}$ and the Lyapunov variable in a standard MI X.

Often the equivalency between a dilated form and its standard counterpart is established by applying the basic Elimination (Projection) Lemma. In the dilated forms the system and Lyapunov variables are separated and there are only product terms of the system matrices and the slack variable(s). Thus, to convexify the multi-objective/robust synthesis problems, a common slack variable is forced for all objectives/corners of uncertainty polytope. This often results in another form of conservatism. Generally, there is no guarantee of improvement using the dilated forms if one only relies on Elimination Lemma to prove the equivalency of a dilated form and its conventional counterpart.

The earliest, and best known, of the matrix dilation results are in the discrete time setting [11, 12], where beside obtaining the dilated form, it is shown that the proposed dilated MIs reduce to the standard ones using

$$(\mathcal{X}, G) = (X, X). \tag{1}$$

The significance of (1) is in multi-objective/robust problems. It guarantees that the solution of the conventional approach is in the feasible solution set of the the dilated form. By seeking different values for Lyapunov matrices of each objective/corner and the slack variable G, the result can be improved. We refer to the approaches that establish the formal connection between the dilated and standard forms in a manner similar to (1) as *constructive* methods.

Although much effort has also been made in the continuous-time case, it is still an open problem, mainly because dilation destroys the convexity in some important cases. Some of the early and convex results are achieved in [20] and [1] via using a form of Elimination Lemma, where a convex search for several important problems such as stability, $\mathcal{H}_2$, D-stability, are obtained. In [5, 6], using a constructive approach, a dilated form which separates A from Lyapunov variable X in the following MI is obtained:

$$AX + XA^T + \delta_1 X + \delta_2 AXA^T + X\Delta^T \Delta X < 0. \tag{2}$$

By assigning different matrixes to A, δ_1, δ_2 and Δ, this general form covers several continuous-time control problems such as stability, $\mathcal{H}_2$ and D-stability. It is also shown that the dilated form recovers (2) by setting $(\mathcal{X}, G) = (X, -a(A - Ia)^{-1}X)$ where a is a positive scalar which appears in the dilated form. This provides formal guarantees that, in multi-objective problems with a common A for all objectives, the dilation-based approach leads to results at least as strong as those of the standard approach. However as indicated in [5], for robust synthesis problems where A is different for each corner there are no such guarantees.

Neither of the two approaches above deal with synthesis inequalities of the $\mathcal{L}_2$-gain (i.e., Bounded Real matrix inequality) or several invariant set determination problems, e.g., energy-to-peak or peak-to-peak. Roughly, these problems result in a constant term in (2). There have been dilated MIs for Bounded Real matrix inequality, as well ([17, 21, 9, 18], among others). However, there seems to be some shortcomings associated with these results. The dilated form introduced in [17] works only for systems with $D_{cl} = 0$, and both [17] and [21] rely on Elimination Lemma to obtain their main results. As a result, as mentioned above (and explained in further details below), the relationship between the original and dilated forms is not clear, and thus there are no guarantees that the dilated forms (in the multi-objective and robust problems) can surpass the original ones. Although the result in [9] is through a constructive method, it exploits a structure that holds only in full state feedback problem. One of the best results obtained so far is in [18] which does not have any of these limitations. However, similar to others [17, 21, 9], it requires an additional scalar variable, entering the MI in a non-convex form. Though such a non-convex form is problematic, the culprit is a scalar variable and it can be addressed by a line search. To avoid this non-convexity, [10] derived a new matrix inequality characterization for Bounded Real matrix inequality by transforming the system into a descriptor form and invoking the results in descriptor system case. As shown in [10] through some examples, less conservative results can be obtained for control synthesis for systems with polytopic uncertainty. However there is no proven guarantee, in general, that this approach is feasible all the time. Also, as point out in [21], the originally avoided matrix coupling reappears in control synthesis because of the terms that include products of the Lyapunov matrix Q_1 and the system matrices A and C in Theorem 2 and A_i and C_i in Corollary 2 of [10].

Here, we present a dilated MI, that can be applied to the Bounded Real matrix inequality. We also extend the results to $\mathcal{H}_2$ and α-stability as well (for extension to the matrix inequalities that are in the invariant set for peak bounded disturbance and the peak norm of a vector over an ellipsoid see [14]). This dilated MI is obtained explicitly through a constructive approach. We show the dilated form recovers the standard MI with the choice of $(\mathcal{X}, G) = (X, X)$. This establishes that in multi-objective/robust synthesis problems, using the proposed dilated form, by seeking different Lyapunov matrices for each objective/corner, the result can be improved. We also rely on a scalar variable that renders the problem non-convex and use line searches to obtain the final results. However, this positive scalar variable belongs to a compact set with known bounds, which makes the line search simpler.

In Section 3, $\mathcal{L}_2$-gain ($\mathcal{H}_\infty$ norm) control design for a system with polytopic uncertainty using the dilated form and the standard form of the Bounded Real Lemma is investigated. We show the problem solved via the dilated form is guaranteed to achieve at least the same performance level as the problem solved with the standard $\mathcal{L}_2$-gain MI. Through a numerical example, the proposed dilated form presented here is compared to the results by the dilated MIs in [17, 21, 9, 18].

Since in the dilated MI obtained here Lyapunov variable appears alone with no multiplication with system matrices, this new form is more promising to be applicable to control structures other than full state feedback. For example, in [14], we used the proposed approach to design a full order dynamic output feedback controller with a multi-objective design requirement of $\mathcal{L}_2$-gain performance while avoiding saturation bounds for a given worst case disturbance bound.

In most results, the development of dilated forms appear somewhat ad-hoc. Recently, [13], showed that a large number of results can be recast as applications of the basic Elimination Lemma, corresponding to different choices of terms. Of course, how one comes up with the dilated forms and whether they recover or indeed improve the standard performance remain open. Generally, an approach that classifies systematic implication and connection is not available. To date, the most general systematic approach has been based on the Full Block S-procedure. Full Block S-procedure has been used extensively in Linear Parameter Varying (LPV) problems in which the control input and measurement output matrices are parameter dependant (e.g. see [16, 4, 22]). In Section 4, we explore the relationship between the dilated forms derived here and Full Block S-procedure.

The system we study has the standard model

$$\left\{ \begin{array}{l} \dot{x}_p = Ax_p + B_1 w + B_2 u \\ z = C_1 x_p + D_{11} w + D_{12} u \\ y = C_2 x_p + D_{21} w \end{array} \right. \tag{3}$$

with the closed-loop

$$\left\{ \begin{array}{l} \dot{x} = A_{cl} x + B_{cl} w \\ z = C_{cl} x + D_{cl} w \end{array} \right. \tag{4}$$

where the details differ in the state feedback and output feedback cases. The transfer function from w to z is called $T_{zw}(s)$.

2 A Dilated Matrix Inequality for $\mathcal{L}_2$-Gain

We will consider the problem of finding the $\mathcal{L}_2$-gain of the system (4).

Lemma 1 ((Bounded Real Lemma [2, 19])) *Consider the closed-loop system (4), A_{cl} is stable and its $\mathcal{L}_2$-gain is less than γ_{con} (i.e., $\|T_{zw}(s)\|_\infty < \gamma_{con}$) if and only if $\bar{\sigma}(D_{cl}) < \gamma_{con}$ and there exists symmetric matrix $X > 0$ such that*

$$\begin{pmatrix} A_{cl}X + XA_{cl}^T & B_{cl} & XC_{cl}^T \\ \star & -\gamma_{con} I & D_{cl}^T \\ \star & \star & -\gamma_{con} I \end{pmatrix} < 0 \tag{5}$$

Matrix X, the Lyapunov variable, enters the equation through the Lyapunov function $V = x^T X^{-1} x$. We have a multiplication between the system matrices (A_{cl}, etc.) and Lyapounv variable X in (5), which causes conservatism in multi-objective/robust synthesis problems by forcing a common Lyapunov variables in order to obtain a unique solution for the controller. Through the following theorem, we show how using matrix dilation one can decouple these variables.

Theorem 1 *The closed-loop system (4) is stable and its $\mathcal{L}_2$-gain is less than γ_{new} (i.e., $\|T_{zw}(s)\|_\infty < \gamma_{new}$) if and only if $\bar{\sigma}(D_{cl}) < \gamma_{new}$, and there exist a positive constant $0 < \epsilon < \frac{1}{2a}$ (where a is an arbitrary positive scalar), and square matrices $\mathcal{X} > 0$ and G which satisfy:*

$$\begin{pmatrix} 2a\mathcal{X} & B_{cl} & 0 & -\mathcal{X} \\ \star & -\gamma_{new}I & D_{cl}^T & 0 \\ \star & \star & -\gamma_{new}I & 0 \\ \star & \star & \star & 0 \end{pmatrix} + \mathcal{Q}^T G \mathcal{P} + \mathcal{P}^T G^T \mathcal{Q} < 0 \tag{6}$$

where $\mathcal{P} = [I \quad 0 \quad 0 \quad -2\epsilon I]$ *and* $\mathcal{Q} = [(A_{cl}^T - aI) \quad 0 \quad C_{cl}^T \quad I]$.

Proof 1 *: The variable $\mathcal{X}$ is the Lyapunov variable, and G is an auxiliary variable (multiplier) introduced for dilation. We show MIs (5) and (6) are equivalent. Suppose that MI (6) holds. Consider the explicit bases of nullspaces of $\mathcal{P}$ and $\mathcal{Q}$*

$$\mathcal{N}_{\mathcal{P}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \frac{1}{2\epsilon}I & 0 & 0 \end{pmatrix}, \quad \mathcal{N}_{\mathcal{Q}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -A_{cl}^T + aI & 0 & -C_{cl}^T \end{pmatrix}$$

By multiplying $\mathcal{N}_{\mathcal{Q}}$ and its transpose from the right and left sides, respectively, and considering that $\mathcal{Q}\mathcal{N}_{\mathcal{Q}} = 0$, the inequality (6) leads to

$$\begin{pmatrix} A_{cl}\mathcal{X} + \mathcal{X}A_{cl}^T & B_{cl} & \mathcal{X}C_{cl}^T \\ \star & -\gamma_{new}I & D_{cl}^T \\ \star & \star & -\gamma_{new}I \end{pmatrix} < 0 \tag{7}$$

Now if we define $\mathcal{X} = X$ and $\gamma_{new} = \gamma_{con}$, (5) holds; i.e., (6) implies (5). Next, multiplying (6) from right and left by $\mathcal{N}_{\mathcal{P}}$ and its transpose respectively gives

$$\begin{pmatrix} (2a - \frac{1}{\epsilon})\mathcal{X} & B_{cl} & 0 \\ \star & -\gamma_{new}I & D_{cl}^T \\ \star & \star & -\gamma_{new}I \end{pmatrix} < 0$$

Thus (6) can have a solution only for $\epsilon < \frac{1}{2a}$.

On the other hand, suppose that (5) holds with $X > 0$. Note that (5) can be rewritten as (7) by defining $X = \mathcal{X}$ and $\gamma_{con} = \gamma_{new}$. Since $\mathcal{X} > 0$, for any $\bar{\epsilon} > 0$ we have

$$\mathcal{R}^T(4\bar{\epsilon}\mathcal{X})^{-1}\mathcal{R} \geq 0 \tag{8}$$

where $\mathcal{R}_1 = [-2\bar{\epsilon}\mathcal{X}(A_{cl}^T - aI) \quad 0 \quad -2\bar{\epsilon}\mathcal{X}C_{cl}^T]$. The left hand side of the inequality above is of order $\bar{\epsilon}$, it is therefore possible to find a sufficiently small $\bar{\epsilon} > 0$ which for any positive $\epsilon \leq \bar{\epsilon}$, the following holds

$$[\text{left side of (7)}] + [\text{left side of (8)}] < 0.$$

Applying the Schur complement to the above inequality, we obtain

$$\begin{pmatrix} \mathbf{He}((A_{cl} - aI)\mathcal{X}) + 2a\mathcal{X} & \star & \star & \star \\ B_{cl}^T & -\gamma_{new}I & \star & \star \\ C_{cl}\mathcal{X} & D_{cl}^T & -\gamma_{new}I & \star \\ -2\epsilon\mathcal{X}(A_{cl} - aI)^T & 0 & -2\epsilon\mathcal{X}C_{cl}^T & -4\epsilon\mathcal{X} \end{pmatrix} < 0$$

By choosing $G = G^T = \mathcal{X}$, this inequality can be written as (6); i.e., satisfaction of (5) leads to a specific choice for the matrices G and $\mathcal{X}$ that satisfy (6), with the same $\mathcal{L}_2$-gain estimate.

Remark 1 *Theorem 1 is similar, structurally, to [9, 18]. However, in [9], the results for the Bounded Real Lemma concern the full state feedback only (There, full state feedback gain, F, is separated from the Lyapunov variable). Since we do not force any special structure on the controller, it can be more promising for use in control structures other than full state feedback (see*

[14], where dynamic output feedback synthesis problem in a multi-objective problem is designed using the dilated form here). The results in [18] and [14] are two special cases of dilated form (6) when a is set to zero and $\frac{1}{2}$, respectively. Although the results in [18] are not restricted to full state feedback control, it results in a search for ϵ over the range $0 < \epsilon < \infty$, which can be an important numerical consideration in multi-objective and robust problems where more than one ϵ is used (see Section 3.1). Using a positive scalar a results in a compact set for the line search over ϵ. For the remainder of this note, we use a positive a and thus $0 < \epsilon < \frac{1}{2a}$

2.1 Extension of the results to $\mathcal{H}_2$ and α-stability

Extension of the proposed dilation to some other important matrix inequalities is straight forward, as many MIs, with some minor changes in the form of system matrices, can be seen as the sub-blocks of (5). Thus the dilated form of these (L)MIs can be deduced from (6). The key feature of these deduced dilated forms is that they all recover the standard matrix inequalities by the choice of $(\mathcal{X}, G) = (X, X)$ which has no dependence on system matrices. As we will see below, this feature is crucial to establish improvement through dilated forms in multi-objective or robust problems. In this section, we give a dilated form for $\mathcal{H}_2$ performance MIs and α-stability.

Standard $\mathcal{H}_2$ performance: A_{cl} is stable with $\|T_{zw}(s)\|_2 < \nu_{con}$ ($D_{cl} = 0$) if and only if there exists $X > 0$ and $Z > 0$ such that

$$\begin{pmatrix} A_{cl}X + XA_{cl}^T & XC_{cl}^T \\ C_{cl}X & -I \end{pmatrix} < 0, \quad \begin{pmatrix} Z & B_{cl}^T \\ \star & X \end{pmatrix} > 0, \qquad trace(Z) < \nu_{con}^2 \tag{9}$$

Theorem 2 *A_{cl} is stable with $\|T_{zw}(s)\|_2 < \nu_{new}$ ($D_{cl} = 0$) if and only if for some $0 < \epsilon < \frac{1}{2a}$ there exist $\mathcal{X} > 0$ and $Z > 0$ such that*

$$\begin{pmatrix} 2a\mathcal{X} & 0 & -\mathcal{X} \\ \star & -I & 0 \\ \star & \star & 0 \end{pmatrix} + \mathbf{He}(\mathcal{Q}^T G \mathcal{P}) < 0, \quad \begin{pmatrix} Z & B_{cl}^T \\ \star & \mathcal{X} \end{pmatrix} > 0, \qquad trace(Z) < \nu_{new}^2 \tag{10}$$

where $\mathcal{P} = [I \quad 0 \quad -2\epsilon I]$ and $\mathcal{Q} = [(A_{cl}^T - aI) \quad C_{cl}^T \quad I]$.

Proof 2 *: We follow the same steps of the proof of the Theorem 1 with $\mathcal{R}$ in (8) being replaced by:*

$$\mathcal{R} = [-2\bar{\epsilon}\mathcal{X}(A_{cl}^T - aI) \quad -2\bar{\epsilon}\mathcal{X}C_{cl}^T]$$

α-stability: For matrix A_{cl} to satisfy α-stability, i.e. $\sigma(A_{cl}) \subset \{\lambda \in \mathbb{C} : \ Re(\lambda) < -\alpha, \alpha > 0\}$, the standard requirement is that there should exist an X such that

$$A_{cl}X + XA_{cl}^T + 2\alpha X < 0. \tag{11}$$

If one replaces $\hat{A}_{cl} \leftarrow A_{cl} + (1/2)\alpha$, then (11) is the sub-block $(1, 1)$ of (5). Therefore, considering (6), the dilated form for this standard LMI is

$$\begin{pmatrix} 2a\mathcal{X} & -\mathcal{X} \\ \star & 0 \end{pmatrix} + \mathbf{He}(\begin{bmatrix} (\hat{A}_{cl} - aI) \\ I \end{bmatrix} G[I \quad -2\epsilon I]) < 0$$

In [14], we also obtain the dilated forms for the invariant set for peak bounded disturbance and peak norm of a vector over an ellipsoid LMI.

Remark 2 *For α-stability and $\mathcal{H}_2$ problem, Ebihara et al. in [5, 6] developed a dilated form without dependency on scalar variable ϵ. But, the choice of $(\mathcal{X}, G)$ to recover the standard results depends on system matrix A_{cl} in (2), $(\mathcal{X}, G) = (X_2, -a(A_{cl} - aI)^{-1}X)$. As indicated in [5], this prevents us from establishing guarantees of improvement through dilated forms, similar to Theorem 3 below, for robust or multi-objective problems with different A_{cl} for each objective. The dilated forms derived here all recover the standard results by setting $(\mathcal{X}, G) = (X, X)$ which has no dependence on system matrices.*

3 $\mathcal{L}_2$-Gain ($\mathcal{H}_\infty$) Control Design for a System with Polytopic Uncertainty

To demonstrate the benefit of the proposed dilated MI, in [14], we solved a full order dynamic output feedback controller design with a multi-objective design requirement of $\mathcal{L}_2$-gain performance while avoiding saturation bounds for a given worst case disturbance bound. Here, we consider the problem of $\mathcal{L}_2$-gain state feedback controller design for a system with polytopic type uncertainty. The standard approach for $\mathcal{L}_2$-gain problem (see e.g., [2]) through the standard Bounded Real LMI (Algorithm 1) and the new dilated MI (Algorithm 2) can be stated as follows

- **Algorithm 1** (Conventional approach): For common $X_i = X > 0$ ($i = 1, 2, ..., N$ where N is the number of the polytope corners), minimize LMI (5) with respect to γ_{con}, for all i corners, i.e.,

$$\begin{pmatrix} \mathbf{He}(A_iX + B_2F) & B_1 & XC_1 + F^TD_{12}^T \\ \star & -\gamma_{con}I & D_{11}^T \\ \star & \star & -\gamma_{con}I \end{pmatrix} < 0$$

- **Algorithm 2** (New approach using new dilated MIs): For $\mathcal{X}_i > 0$, and common $G_i = G$ ($i = 1, 2, ..., N$), minimize γ_{new} in (6) for all i corners, i.e.,

$$\begin{pmatrix} \mathcal{X}_i + \mathbf{He}(A_iG + B_2F - \frac{1}{2}G) & B_1 & G^TC^T + F^TD_{12}^T & -\mathcal{X}_i + G^T - 2\epsilon_i(A_iG + B_2F - \frac{1}{2}G) \\ \star & -\gamma_{new}I & D_{11}^T & 0 \\ \star & \star & -\gamma_{new}I & -2\epsilon_i(CG + D_{12}F) \\ \star & \star & \star & -2\epsilon_i(G^T + G) \end{pmatrix} < 0$$

Here we used $a = \frac{1}{2}$.

These inequalities are the direct result of replacing the closed loop matrices and calling $F = KX$ in (5) and $F = KG$ in (6) respectively, where K is the full state feedback gain. In Algorithm 1 to make the MI set convex we are forced to us common Lyapunov matrix for all the corners. While, in Algorithm 2, we can use different Lyapunov matrices.

Theorem 3 ((Guarantee of Improvement)) *Algorithm 2 with a common auxiliary variable, G, but with different Lyapunov variables, always achieves an upper bound estimate for the $\mathcal{L}_2$-gain that is less than or equal to the $\mathcal{L}_2$-gain performance estimate achieved by Algorithm 1.*

Proof 3 *: If Algorithm 1 is solved, then Theorem 1 implies that there is a positive $\bar{\epsilon}_i$ such that for any $\epsilon_i < \bar{\epsilon}_i$ by taking $\mathcal{X}_i = G = G^T = X$ and $\gamma_{new} = \gamma_{con}$, we can satisfy (6) with the same closed-loop system derived by solving Algorithm 1. Therefore, any solution of Algorithm 1 can be achieved by Algorithm 2, for small enough ϵ_i's, without exploiting the ability to use different Lyapunov matrices, which could only improve the results.*

Remark 3 *To reduce the computational cost of line search over ϵ_i's, one can use a common ϵ without losing the guarantee of improvement of Theorem 3 ($\epsilon < \epsilon_i$ $\forall i$).*

3.1 Numerical example: polytopic uncertainty

In this section, we investigate Algorithms 1 and 2 of Section 3 through a numerical example. Consider the system below, taken from [17]

$$\left(\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \end{array}\right) = \left(\begin{array}{cccc|c|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & k & -f & f & 0 & 1 \\ k & -k & f & -f & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 \end{array}\right) \tag{12}$$

where $k \in [0.09 \quad 0.6]$ and $f \in [0.0038 \quad 0.04]$. This example is studied in [21] and compared with the results of [17] and [10]. We call these tests $T2$, $T3$ and $T4$, respectively. Here, we solve this example through Algorithm 2, denoted as $T7$. To complete the comparison, we add the results by using the MIs of [9] as $T5$, and [18] as $T6$. The result of Algorithm 1 is called $T1$. The best γ obtained are as follows:

T1	T2	T3	T4	T5	T6	**T7**
[2]	[21]	[17]	[10]	[9]	[18]	Her
2.044	1.867	1.924	2.044	1.638	1.548	1.548

All of the dilated forms are giving better or same result compared to the result of Algorithm 1, $T1$. The best result is obtained through test $T6$ for $\epsilon = 0.071$, and $T7$ for $\epsilon = 0.035$.

4 Full Block S-procedure Approach for Matrix Dilation

In this section we discuss connections between the dilated form (6) and the Full Block S-procedure (see Appendix A for a short review of Full Block S-procedure). Full Block S-procedure has been used extensively in Linear Parameter Varying (LPV) problems in which the control input and measurement output matrices are parameter dependent (e.g. see [16, 4, 22]). Here, we first review the robust $\mathcal{L}_2$-gain ($\mathcal{H}_\infty$) characterization of [16] for parameter dependent systems using the Full Block S-procedure.

Consider a closed-loop system (4), with time varying parametric uncertainties collected in matrix Δ. Using the Linear Fractional Transformation (LFT) results (e.g., see [23]), any such interconnected systems may be rearranged to fit the general framework below:

$$\begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} = \begin{pmatrix} \widehat{A} & \widehat{B}_1 \\ \widehat{C}_1 & \widehat{D}_1 \end{pmatrix} + \begin{bmatrix} \widehat{B}_2 \\ \widehat{D}_{12} \end{bmatrix} \triangle (I - \widehat{D}_2 \triangle)^{-1} \begin{bmatrix} \widehat{C}_2 & \widehat{D}_{21} \end{bmatrix} \tag{13}$$

As mentioned above, one way to obtain the $\mathcal{L}_2$-gain (to be precise, an upper bound estimate) of (13) is via using the Bounded Real Lemma (1), and form (5). Then there would be a multiplication between the uncertainty matrix Δ and the Lyapunov variable X, resulting in imposing common X for robust synthesis problems. Another way to solve this problem is through the Full Block S-procedure as proposed in [16]. In the following, we review how the Full Block S-procedure is used to obtain an equivalent matrix inequality representation for (5) with X and Δ decoupled. To obtain the decoupled form for (5), the following set of the multipliers is defined in [16]:

$$\widetilde{\mathcal{P}} = \left\{ \widetilde{P} = \widetilde{P}^T \in \mathbb{R}^{(k+1)\times(k+l)} : \begin{pmatrix} I \\ \triangle^T \end{pmatrix}^T \widetilde{P} \begin{pmatrix} I \\ \triangle^T \end{pmatrix} \geq 0 \right\} \tag{14}$$

where any such multiplier is partitioned as

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_{12} & \widetilde{P}_{12} \\ \widetilde{P}_{12}^T & \widetilde{P}_{22} \end{pmatrix} \text{ comformable to } \begin{pmatrix} I \\ \triangle^T \end{pmatrix}$$

Theorem 4 ([16]) *The interconnection (13) is well-posed and the symmetric matrix $X > 0$ satisfies (5) iff there exist $\mathcal{X} > 0$ and $\widetilde{P} \in \widetilde{\mathcal{P}}$ such that*

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & \mathcal{X} & 0 & 0 & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \widetilde{P}_{11} & \widetilde{P}_{12} & 0 & 0 \\ 0 & 0 & \widetilde{P}_{12}^T & \widetilde{P}_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma \end{array}\right) \left(\begin{array}{ccc} \widehat{A}^T & \widehat{C}_2^T & \widehat{C}_1^T \\ I & 0 & 0 \\ \hline \widehat{B}_2^T & \widehat{D}_2^T & \widehat{D}_{12}^T \\ 0 & I & 0 \\ \hline \widehat{B}_1^T & \widehat{D}_{21}^T & \widehat{D}_1^T \\ 0 & 0 & I \end{array}\right) < 0 \tag{15}$$

Several choices are possible for the class of multipliers, e.g., the use of constant block-diagonal multipliers for robust control problems has been exploited in [7]. In [4] the following $\widetilde{P}$ which depends on the $\triangle$ with arbitrary G_0 and G_1 is used as the multiplier:

$$\widetilde{P} = \begin{pmatrix} G_0\triangle^T + \triangle G_0^T & -G_0 - \triangle G_1^T \\ -G_0^T - G_1\triangle^T & G_1 + G_1^T \end{pmatrix} \tag{16}$$

This class of multipliers renders the left hand side of the inequality in (14) identically zero, therefore, to obtain the decoupled equivalent form for (5), one only needs to guarantee (15). Note that using this multiplier, (15) is an equivalent form for (5) with the Lyapunov variable X decoupled from uncertainty matrix Δ.

If one looks at the Δ as the part of the dynamics that is intended to be separated from the Lyapunov variable, Full Block S-procedure can be used to obtain dilated matrix inequalities. The main challenge here is to represent the system dynamics in the LFT form (13). In the following, we show that by a particular choice of the (non-unique) LFT form (13), our proposed dilated form (6) becomes equivalent to the results obtained via Full Block procedure with fewer free parameters. Consider

$$\begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} = \begin{pmatrix} a\,I & B_{cl} \\ 0 & D_{cl} \end{pmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} A_{cl} - a\,I & 0 \\ C_{cl} & 0 \end{pmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

Comparing this with the LFT form (13), we obtain:

$$\begin{cases} \widehat{A} = a\,I, \quad \widehat{B}_1 = B_{cl} \quad \widehat{C}_1 = 0, \quad \widehat{D}_1 = D_{cl} \\ \widehat{B}_2 = [I \quad 0], \quad \widehat{D}_{12} = [0 \quad I], \quad \widehat{D}_2 = 0 \\ \triangle = \begin{pmatrix} A_{cl} - a\,I & 0 \\ C_{cl} & 0 \end{pmatrix}, \ \widehat{C}_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \ \widehat{D}_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases} \tag{18}$$

Next, let us partition G_0 and G_1 in (16) as follows

$$G_0 = \begin{pmatrix} G_{01} & G_{02} \\ G_{03} & G_{04} \end{pmatrix}, \quad G_1 = \begin{pmatrix} G_{11} & G_{12} \\ G_{13} & G_{14} \end{pmatrix} \ \text{both comformable to} \ \begin{pmatrix} A_{cl}^T - a\,I & C_{cl}^T \\ 0 & 0 \end{pmatrix}$$

Then, using the multiplier $\widetilde{P}$ with structure (16) and the LFT form with matrices given in (18), and applying Schur complement, one can expand (15) as

$$\begin{pmatrix} 2a\,\mathcal{X} + \mathbf{He}(G_{01}(A_{cl}^T - aI)) & \mathcal{X} - G_{01} - (A_{cl} - aI)G_{11}^T & -G_{02} - (A_{cl} - aI)G_{13}^T & G_{01}C_{cl}^T + (A_{cl} - aI)G_{03}^T & B_{cl} \\ \star & G_{11} + G_{11}^T & G_{12} + G_{13}^T & -G_{03}^T - G_{11}C_{cl}^T & 0 \\ \star & G_{13} + G_{12}^T & G_{14} + G_{14}^T & -G_{04}^T - G_{13}C_{cl}^T & 0 \\ \star & \star & \star & G_{03}C_{cl}^T + C_{cl}G_{03}^T - \gamma & D_{cl} \\ \star & \star & \star & \star & -\gamma \end{pmatrix} < 0 \tag{19}$$

In synthesis problems the closed-loop matrices A_{cl} etc. contain the unknown controller matrices. Therefore, the above matrix inequality, because of the the terms such as $A_{cl}G_{01}^T$, $A_{cl}G_{11}^T$, $A_{cl}G_{13}^T$, $A_{cl}G_{03}^T$, $C_{cl}G_{11}^T$, etc., is not convex. In order to obtain a numerical solution using efficient numerical solvers, we need a convex form. To convexify, similar to [4], we use the following structure for G_0 and G_1.

$$G_0 = \begin{pmatrix} G & G_{02} \\ 0 & G_{04} \end{pmatrix}, \quad G_1 = \begin{pmatrix} \alpha G & G_{12} \\ 0 & G_{14} \end{pmatrix}.$$

Both G_{11} and G_{01} are multiplied by A_{cl}^T and C_{cl}^T. For example in case of state feedback with control gain K, we have

$$\begin{aligned} A_{cl}G_{01}^T = AG_{01}^T + B_2KG_{01}^T, \quad C_{cl}G_{01}^T = C_1G_{01}^T + D_{12}KG_{01}^T \\ A_{cl}G_{11}^T = AG_{11}^T + B_2KG_{11}^T, \quad C_{cl}G_{11}^T = C_1G_{11}^T + D_{12}KG_{11}^T \end{aligned}$$

These inequalities are often made convex by using new (intermediate) variables such as $F = KG_{01}^T$ in the state feedback. In order to obtain the same unique controller gain, G_{11} should either be the same as G_{01} or at most a scalar multiple of it (in which case the scalar enters as a decision variable). Here, G_{03} and G_{13}, which also multiply C_{cl}^T and A_{cl}^T similar to G_{01} and G_{11}, are not square matrices and, to be able to obtain the controller gain, are set to zero. Inequality (19) then becomes

$$\begin{pmatrix} 2a\,\mathcal{X} + \mathbf{He}(G(A_{cl}^T - aI)) & \mathcal{X} - G - \alpha(A_{cl} - aI)G^T & -G_{02} & GC_{cl}^T & B_{cl} \\ \star & \alpha(G + G^T) & G_{12} & -\alpha GC_{cl}^T & 0 \\ \star & G_{12}^T & G_{14} + G_{14}^T & -G_{04}^T & 0 \\ \star & \star & \star & -\gamma & D_{cl} \\ \star & \star & \star & \star & -\gamma \end{pmatrix} < 0 \tag{20}$$

In case of dynamics full order output feedback controller, using an approach similar to [20], one can obtain a characterization of (20) which is convex modulo a line search for α (see [14] for details).

Remark 4 *With a congruent transformation Diag*$[I \quad -I \quad I \quad I \quad I]$*, for* $\alpha = -2\varepsilon$*, and*

$$G_0 = \begin{pmatrix} G^T & 0 \\ 0 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} \alpha G^T & 0 \\ 0 & G_{14} \end{pmatrix}, \tag{21}$$

the dilated form introduced in Theorem 1 becomes a special case of the dilated form (20) obtained using Full Block S-procedure approach. Note that $G_{14} + G_{14}^T$ *appears only in the third diagonal entry with the rest of the row and column elements corresponding to it being zero. Therefore, the solution is independent of* G_{14}*.*

Next, we show that G_{12}, G_{02}, G_{14}, and G_{04} are redundant parameters in (20); and using the special structure (21) to arrive at (15) is without any loss of generality. Note that we can write (20) as

$$\begin{pmatrix} 2a\,\mathcal{X} + \mathbf{He}(G(A_{cl}^T - aI)) & \star & \star & \star & \star \\ \mathcal{X} - G^T - \alpha G(A_{cl}^T - aI) & \alpha(G + G^T) & \star & \star & \star \\ 0 & \star & 0 & \star & \star \\ C_{cl}G^T & -\alpha C_{cl}G^T & 0 & -\gamma & \star \\ B_{cl}^T & 0 & 0 & D_{cl}^T & -\gamma \end{pmatrix} + \mathcal{Q}^T\mathcal{G}\mathcal{P} + \mathcal{P}^T\mathcal{G}^T\mathcal{Q} < 0 \tag{22}$$

where $\mathcal{Q}$ and $\mathcal{P}$ are

$$\mathcal{G}^T = \begin{bmatrix} -G_{02}^T & G_{12}^T & G_{14}^T & -G_{04}^T \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0 & 0 & I & 0 & 0 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

Bases of nullspaces of $\mathcal{P}$ and $\mathcal{Q}$ are

$$\mathcal{N}_{\mathcal{P}} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \mathcal{N}_{\mathcal{Q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix}$$

Applying the Elimination Lemma on (20), then

$$\mathcal{N}_{\mathcal{P}}^T[\text{right side of (20)}]\mathcal{N}_{\mathcal{P}} < 0$$

results in

$$\begin{pmatrix} 2a\,\mathcal{X} + \mathbf{He}(G(A_{cl}^T - aI)) & \star & \star & \star \\ \mathcal{X} - G^T - \alpha G(A_{cl}^T - aI) & \alpha(G + G^T) & \star & \star \\ C_{cl}G^T & -\alpha C_{cl}G^T & -\gamma & \star \\ B_{cl}^T & 0 & D_{cl}^T & -\gamma \end{pmatrix} < 0 \tag{23}$$

while the trivial $-\gamma < 0$ is resulted from

$$\mathcal{N}_{\mathcal{Q}}^T[\text{right side of (20)}]\mathcal{N}_{\mathcal{Q}} < 0.$$

This means that (23) and (20) are equivalent or, in another word, the best γ obtained subject to (20) is independent of G_{02}, G_{04}, G_{12} and G_{14}. For the example of Section 3.1, the particular LFT form suggested in [4] gives us the same numerical results as our technique here, albeit with more search variables.

In summary, using the LFT in (17) (or (18)), after performing the simplifications required for convexity, the Full Block S-procedure yields a result that is equivalent to our result, though with more search variables. The Full Block S-procedure, as mentioned in [4] can be extended to output feedback synthesis case. Given the structure and the seemingly non-performing extra search variables, the extension could be somewhat cumbersome, compared to extensions of the results presented here to output feedback case (see [14]).

Finally, as discussed in [4], the non-unique LFT representation remains an open issue. The generality of the Full Block S-Procedure might offer the chance to interpret several of the available dilated forms as the result of different choices for the underlying LFT form used to model the dynamics, or the structure of the multipliers used. This, in turn, could provide some insight to the underlying structural properties of the dilations used. This is left to future work.

5 Conclusions

We present dilated matrix inequalities for Bounded Real MI, which can easily be extended to other common stability and performance MIs such as $\mathcal{H}_2$ or α-stability. The dilated form here is obtained without imposing any structure on the closed-loop matrices. As a result, this dilated form can be used in both state feedback and dynamic output feedback controller design design in multi-objective problems. Moreover, we show that any solution of the Bounded Real MI can be recovered by the dilated form by letting $(\mathcal{X}, G) = (X, X)$, similar to results in discrete time dilated forms in [11, 12]. In multi-objective and robust control synthesis problems, despite the common auxiliary variable G, we show that in our approach the controller design via dilated forms can always recover the solution of the conventional approach. This affords the opportunity for improved results by allowing different Lyapunov matrices for different objectives or corners of uncertainty.

We also draw parallels between the proposed dilated form and the results based on the more systematic approach generated by Full Block S-procedure. Natural extension to this part of our work would be to look into the connection between different dilated forms and Full Block S-procedure. The connection could be through a different LFT form or different choice of multipliers. Perhaps a unified tool developed via the S-procedure framework can clarify the source of the possible conservatism in dilated forms.

A Review of Full Block S-procedure

In the following, we review the full-block S-procedure as presented in [16]. Suppose $\mathcal{S}$ is a subset of $\mathbb{R}^n$, $T \in \mathbb{R}^{l\times n}$ is full row rank matrix and $\mathbf{U} \subset \mathbb{R}^{k\times l}$ is a compact set of matrices of full row rank. Also,

$$\begin{aligned} \mathcal{S}_U &:= \mathcal{S} \cap \ker(UT) = \{x \in \mathcal{S} :\ UTx = 0\} \\ &= \{x \in \mathcal{S} :\ Tx \in \ker(U)\} \end{aligned} \tag{24}$$

where $U \in \mathbf{U}$.

Suppose $N \in \mathbb{R}^{n\times n}$ is a fixed symmetric matrix. The goal is to render the implicit negativity condition

$$\forall U \in \mathbf{U} : \quad N < 0 \;\; \textit{on} \;\; \mathcal{S}_U$$

explicit. We want to relate this property, under certain technical hypotheses, to the existence of a multiplier P that satisfies

$$N + T^T PT < 0 \;\; on \;\; \mathcal{S} \;\; and \;\; P > 0 \;\; on \;\; ker(U)$$

for all $U \in \mathbf{U}$

The required technical condition will be related to a certain well-posedness property; here it amounts to the complementarity of the subspace $\mathcal{S}_U$ to a fixed subspace $\mathcal{S}_0 \subset \mathcal{S}$ that is sufficiently large. Moreover, the quadratic form N is supposed to be nonnegative on this subspace. To be precise, we require

$$dim(\mathcal{S}_0) < k \;\; and \;\; N \geq 0 \;\; on \;\; \mathcal{S}_0.$$

Theorem 5 (Full Block S-procedure [16]) *The condition*

$$\forall U \in \mathbf{U} : \mathcal{S}_U \cap \mathcal{S}_0 = 0, \quad N < 0 \;\; on \;\; \mathcal{S}_U \tag{25}$$

holds iff there exists a matrix that satisfies

$$\forall U \in \mathbf{U} : \begin{cases} N + T^T PT < 0 \;\; \text{on} \;\; \mathcal{S} \\ P \geq 0 \;\; \text{on} \;\; \ker(U) \end{cases} \tag{26}$$

In the intended application below, $\mathcal{S}$ is an unperturbed system, T picks the interconnection variables that are constrained by the uncertainties, the elements of $U \in \mathbf{U}$ define kernel representations of the possible uncertainties, and $\mathcal{S}_U$ is the uncertain system.

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