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Accelerated Average Consensus Algorithm Using Outdated Feedback

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***Abstract*— This paper examines accelerating the well-known Laplacian average consensus algorithm by breaking its conventional delay-free input into two weighted parts and replacing one of these parts by an outdated feedback. We determine for what weighted sum there exists a range of time delay that leads to increase in the rate of convergence of the algorithm. For such weights, using the Lambert W function, we obtain the rate increasing range of the time delay and also the maximum reachable rate and its corresponding maximizer delay. We also specify what combinations of the current and an outdated feedback increase the rate of convergence without increasing the control effort of the agents. Lastly, we determine the optimum combination of the current and the outdated feedback weights to achieve maximum increase in the rate of convergence without increasing the control effort. We demonstrate our results through a numerical example.**

I. Introduction

In a network of N agents each endowed with a constant reference input $\mathsf{r}^i \in \mathbb{R}$, the average consensus problem consists of designing a distributed inter-neighbor interaction policy for each agent $i \in \{1, \dots, N\}$ such that a local agreement state x^i asymptotically converges to $\frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j$. The solution to this problem is of significance in various multi-agent applications such as sensor fusion [1]–[3], robot coordination [4], formation control [5] and distributed estimation [6]. The widely adopted distributed solution for this problem is the simple Laplacian algorithm [7]–[10]. In this algorithm, each agent initializes its local first order integrator dynamics with its local reference value and uses the sum of weighted difference between its local state and those of its neighbors as the input to drive its dynamics. Accelerating the convergence of this algorithm results in fast agreement in the network and is of prime interest in practice. For a connected network with undirected communication, it is well understood that the rate of convergence of the Laplacian average consensus algorithm is defined by the smallest non-zero eigenvalue of the Laplacian matrix [7], which is also a measure of connectivity of the graph [11]. Given this connection, various efforts such as optimal adjacency weight selection for a given topology by maximizing the smallest non-zero eigenvalue of the Laplacian matrix [9], [12] or rewiring the graph to create topologies such as small world network [13], [14], which are known to have high connectivity, have been proposed in the literature. In this paper, we take a different approach, and aim to exploit time delayed feedback to improve the convergence of the average consensus algorithm without increasing the control effort over a given network topology. Our method can be applied in conjunction with topology designs to maximize the effect.

Intuition links time delay in dynamical systems to sluggishness in system performance. However, some literature points to the positive effect of time delay on increasing stability margin and rate of convergence of time-delayed systems, see e.g., [15]–[22]. In particular, the evidence of positive effect of time-delayed feedback on increasing the rate of convergence of Laplacian dynamics is presented in [18], [22]. In [22], the increase of rate of convergence in a Laplacian average consensus algorithm that uses only an out-dated feedback is demonstrated, but without characterizing the effect of delay on the control effort. In [18] combination of current and out-dated feedback is utilized to increase the convergence rate. However, the size of current and out-dated feedback are considered equal which restricts the benefits of exploiting out-dated feedback.

Here, we introduce a gain in the outdated feedback of the Laplacian average consensus algorithm and examine thoroughly its effect to obtain a specific range of delay such that higher rate of convergence is achieved without increasing the control effort of the agents.We describe in full the convergence rate in terms of Lambert W function. The exact value of the rate of convergence of time-delayed LTI systems is specified in [23]–[25] using the Lambert W function. By relying on this result and the properties of the Lambert W function (see [26], [27]), we start our study by characterizing fully the variation of the rate of convergence of a scalar system with delay. Our results specify (a) for what values of the system parameters the rate of convergence in the presence of delay can increase, (b) the exact values of delay for which the rate of convergence increases, and (c) the optimum value of τ corresponding to the maximum rate of convergence in the presence of delay. Next, we apply our results to analyze the positive effect of adding an outdated disagreement feedback to accelerate the rate of convergence of the Laplacian average consensus algorithm. Unlike the method of [18], our technique allows adding a gain to the outdated disagreement feedback to study the effect of the relative size of the outdated and immediate feedback terms. Then, we determine for what values of the outdated feedback gain, the average consensus algorithm can have a higher rate of convergence in the presence of time delay. We also determine a specific region for the gain such that the maximum control effort by the agents does not increase but the convergence rate improves. We show that the optimum value of gain is determined by considering a trade-off between the performance (maximum convergence rate) and the robustness of the algorithm. Due to space limitation,

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for some of our results we only provide a sketch of the proof. The Complete proofs will appear elsewhere.

II. Notations and Preliminaries

We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the set of real, positive real, non-negative real, integer, and complex numbers, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$, $\mathrm{Im}(s)$ represent its real and imaginary parts, respectively. Moreover, $|s| = \sqrt{\mathrm{Re}(s)^2 + \mathrm{Im}(s)^2}$ and $\arg(s) = \mathrm{atan2}(\mathrm{Im}(s), \mathrm{Re}(s))$. For a measurable locally essentially bounded function $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we define $|\mathbf{u}|_\infty = \sup\{\|\mathbf{u}(t)\|_\infty,\ t \geq 0\}$. For a matrix $\mathbf{A}$, its i^{th} row is denoted by $[\mathbf{A}]_i$.

Lambert W function specifies the solutions of $s\,\mathrm{e}^s = z$ for a given $z \in \mathbb{C}$, i.e., $s = W(z)$. It is a multivalued function with infinite number of solutions denoted by $W_k(z)$, $k \in \mathbb{Z}$, where W_k is called the k^{th} branch of W function. For any $z \in \mathbb{C}$, $W_k(z)$ can readily be evaluated in Matlab or Mathematica. Below are some of the natural properties of the Lambert W function, which we use (see [26], [27]),

$$\lim_{z \to 0} W_k(z)/z = 1, \tag{1a}$$

$$\mathrm{d}\,W_k(z)/\mathrm{d}\,z = 1/(z + \mathrm{e}^{W_k(z)}), \qquad \text{for } z \neq 1/\mathrm{e}, \tag{1b}$$

for $k \in \mathbb{Z}$. For any $z \in \mathbb{R}$, the value of all the branches of the Lambert W function except for the branch 0 and the branch -1 are complex, i.e., they have non-zero imaginary part. Moreover, the zero branch satisfies $W_0(-1/\mathrm{e}) = -1$, $W_0(0) = 0$ (see Fig. 1) and

$$W_0(z) \in \mathbb{R}, \qquad z \in [-1/\mathrm{e}, \infty), \tag{2a}$$

$$\mathrm{Im}(W_0(z)) \in (-\pi, \pi)\backslash\{0\}, \qquad z \in \mathbb{C}\backslash[-1/\mathrm{e}, \infty), \tag{2b}$$

$$\mathrm{Re}(W_0(z)) > -1, \qquad z \in \mathbb{R}\backslash\{-1/\mathrm{e}\}, \tag{2c}$$

Lemma 2.1 (Maximum real part of Lambert W function [26]): *For any $z \in \mathbb{C}$, the following holds*

$$\mathrm{Re}(W_0(z)) \geq \max\big\{\,\mathrm{Re}(W_k(z))|\ k \in \mathbb{Z}\backslash\{0\}\big\}. \tag{3}$$

The equality holds between branch 0 and -1 over $z \in (-\infty, -\frac{1}{\mathrm{e}})$ where we have $\mathrm{Re}(W_0(z)) = \mathrm{Re}(W_{-1}(z))$. □

Lemma 2.2 ($W_0(\mathsf{x})$ is an increasing function of $x \in \mathbb{R}_{>0}$): *For any $\mathsf{x}, \mathsf{y} \in \mathbb{R}_{>0}$ if $\mathsf{x} < \mathsf{y}$, then $W_0(\mathsf{x}) < W_0(\mathsf{y})$.*

Proof: The proof follows from the fact that for $\mathsf{x} \in \mathbb{R}_{>0}$, $W_0(\mathsf{x}) \in \mathbb{R}_{>0}$. Therefore, $\frac{\mathrm{d}\,W_0(\mathsf{x})}{\mathrm{d}\,\mathsf{x}} = \frac{1}{\mathsf{x} + \mathrm{e}^{W_0(\mathsf{x})}} > 0$. ■

We follow [28] to define our graph related terminologies and notations. In a network of N agents, we model the inter-agent interaction topology by the undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{A})$ where $\mathcal{V} = \{1, \cdots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathbf{A} = [a_{ij}]$ is the adjacency matrix of the graph. Recall that $a_{ii} = 0$, $a_{ij} \in \mathbb{R}_{>0}$ if $j \in \mathcal{V}$ can send information to agent $i \in \mathcal{V}$, and zero otherwise. Moreover, a graph is undirected if the connection between the nodes is bidirectional and $a_{ij} = a_{ji}$ if $(i,j) \in \mathcal{E}$. Finally, an undirected graph is connected if there is a path from every agent to every other agent in the network (see e.g. Fig. 2). $\mathbf{L} = \mathrm{Diag}(\mathbf{A}\mathbf{1}_N) - \mathbf{A}$ is the Laplacian matrix of the graph

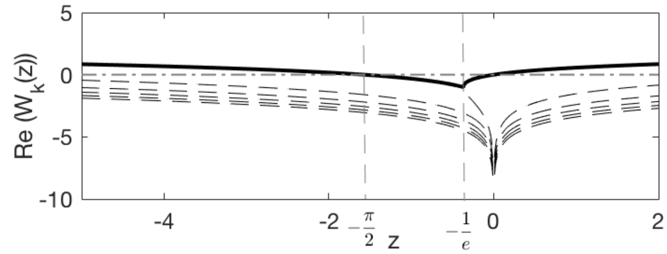
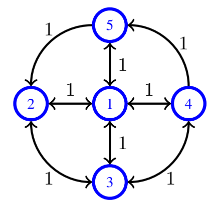


Fig. 1: The solid thick line shows 0 branch while the thin dashed lines show the other branches in $k = \{-5, -4, \cdots, 4, 5\}$ (some of the branches overlap).



$$\mathbf{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Fig. 2: A connected graph of 5 nodes.

$\mathcal{G}$. The Laplacian matrix of a connected undirected graph is a symmetric positive semi-definite matrix that has a simple $\lambda_1 = 0$ eigenvalue, and the rest of its eigenvalues satisfy $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N$. Moreover, $\mathbf{L}\mathbf{1}_N = \mathbf{0}$. Since $\mathbf{L}$ of a connected undirected graph is a symmetric and real matrix, its normalized eigenvectors $v_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N, v_2, \cdots, v_N$ are mutually orthogonal. Moreover for

$$\mathbf{T} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & \mathbf{R} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} v_2 & \cdots & v_N \end{bmatrix} \tag{4}$$

we have $\mathbf{T}^\top\mathbf{L}\mathbf{T} = \mathbf{\Lambda} = \mathrm{Diag}(0, \lambda_2, \cdots, \lambda_N)$.

III. Problem Definition

We consider the static average consensus problem over a multi-agent system where the agreement dynamics of each agent is given by $\dot{x}^i(t) = u^i$ with u^i as the control input of agent $i \in \mathcal{V}$. When the graph topology $G(\mathcal{V}, \mathcal{E}, \mathbf{A})$ is an undirected connected, the algorithm

$$\dot{x}^i(t) = u^i = -\alpha \sum\nolimits_{j=1}^{N} a_{ij}(x^j(t) - x^i(t)), \quad x^i(0) = \mathsf{r}^i, \tag{5}$$

for $i \in \mathcal{V}$, $\alpha \in \mathbb{R}_{>0}$, is proposed as a solution for the average consensus algorithm, i.e., $\lim_{t\to\infty} x^i = \mathsf{x}^{\text{avg}}(0) = \frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j$. The convergence of (5) is exponential with the rate of convergence $\rho_0 = \alpha\lambda_2$ (for details see [10]). In this paper we alter algorithm (5) as follows (compact representation),

$$\dot{\mathbf{x}}(t) = -\alpha\,(1 - \mathsf{k})\,\mathbf{L}\,\mathbf{x}(t) - \alpha\,\mathsf{k}\,\mathbf{L}\,\mathbf{x}(t - \tau), \tag{6a}$$

$$x^i(\eta) = \phi^i(\eta) = 0 \qquad \eta \in [-\tau, 0], \qquad i \in \mathcal{V}, \tag{6b}$$

for $t \in \mathbb{R}_{\geq 0}$, where $\mathsf{k} \in \mathbb{R}/\{0\}$; for $\mathsf{k} = 0$ we recover the original algorithm (5). Our objective in this paper is to show that by splitting the disagreement feedback into a current $-\alpha\,(1 - \mathsf{k})\,\mathbf{L}\,\mathbf{x}(t)$ and an out-dated $-\alpha\,(1 - \mathsf{k})\,\mathbf{L}\,\mathbf{x}(t - \tau)$

components, it is possible to increase the rate of convergence without increasing the control effort. Specifically, we show that for all $\mathsf{k} \in \mathbb{R}_{>0}$, there always exists a range of delay $(0, \tilde{\tau}_{\mathsf{k}})$ such that $\rho_{\tau,\mathsf{k}} > \rho_0$ for any $\tau \in (0, \tilde{\tau}_{\mathsf{k}})$. Here $\rho_{\tau,\mathsf{k}}$ is the rate of convergence of the modified algorithm (6) in the presence of delay for a given $\mathsf{k} \in \mathbb{R}_{>0}$. We show however that only for $\mathsf{k} \in (0, 1]$ we can guarantee $|\mathbf{u}_{\tau,\mathsf{k}}|_\infty \leq |\mathbf{u}_{0,0}|_\infty$, for $\tau \in (0, \tilde{\tau}_{\mathsf{k}})$. In what follows, we also investigate what the maximum value of $\rho_{\tau,\mathsf{k}}$ and the corresponding maximizer $\tau^\star_{\mathsf{k}} \in (0, \tilde{\tau}_{\mathsf{k}})$ are for a given $\mathsf{k} \in \mathbb{R}_{>0}$.

For convenience in our study, we implement the change of variable $\mathbf{z}(t) = \mathbf{T}^\top(\mathbf{x}(t) - \mathsf{x}^{\text{avg}}(0)\mathbf{1}_N)$ (recall (4)) to write (6) in the following equivalent form

$$\dot{z}_1(t) = 0, \qquad z_1(0) = 0, \tag{7a}$$
$$\dot{z}_i(t) = -\alpha(1-\mathsf{k})\lambda_i\, z_i(t) - \alpha\mathsf{k}\,\lambda_i z_i(t-\tau), \tag{7b}$$
$$z_i(0) = [\mathbf{T}^\top \mathbf{x}(0)]_i, \quad z_i(\eta) = 0 \quad \eta \in [-\tau, 0), \tag{7c}$$

for $i \in \{2, \cdots, N\}$. Since (7) is a set of scalar dynamics, in the proceeding section we develop a set of preliminary results that characterize the effect of delay on a class of scalar time-delayed systems with a structure similar to (7b). We then use these results to carry out our main study in Section V.

IV. Preliminaries: A scalar time-delayed system

Consider the scalar linear time-delayed system

$$\begin{aligned} \dot{x}(t) &= \alpha\, x(t-\tau) + \beta\, x(t), \qquad && t \in \mathbb{R}_{\geq 0}, \\ x(t) &\in \mathbb{R}, && t \in [-\tau, 0], \end{aligned} \tag{8}$$

where state $x(t) \in \mathbb{R}$ at time t, $\tau \in \mathbb{R}_{>0}$ denotes the *time delay* and $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$ are the known system parameters satisfying $\alpha + \beta < 0$. For a given $\tau \in \mathbb{R}_{>0}$, (8) is exponentially stable with the rate of ρ_τ if and only if there exists a $c_\tau \in \mathbb{R}_{>0}$ and an $\rho_\tau \in \mathbb{R}_{>0}$ such that for the given initial condition in (8) the solution satisfies

$$|x(t)| \leq c_\tau\, \mathrm{e}^{-\rho_\tau t} \sup_{t \in [-\tau, 0]} |x(t)|, \quad t \in \mathbb{R}_{\geq 0}. \tag{9}$$

For any given $\tau \in \mathbb{R}_{>0}$, the exponential stability of system (8) is guaranteed when all the roots of its characteristic equation $\mathcal{F}(s) = s - \alpha\, e^{-\tau\, s} - \beta = 0$, which are specified by (see [23])

$$\{s \in \mathbb{C} \,|\, s = \frac{1}{\tau} W_k(\alpha\, \tau\, \mathrm{e}^{-\tau\beta}) + \beta, \quad k \in \mathbb{Z}\} \tag{10}$$

are located strictly in the left hand side of the complex plane [29]. The *critical* value of delay $\bar{\tau} \in \mathbb{R}_{>0}$ beyond which the system is unstable is obtained from the smallest value of $\tau \in \mathbb{R}_{>0}$ for which the rightmost root of the characteristic equation is on the imaginary axis. Using this approach the admissible delay range for delayed system (8) is given as follows.

Theorem 4.1 (Admissible delay bound for (8) [29, Proposition 3.15]): *The following assertions hold for system* (8) *with* $\alpha \in \mathbb{R}\backslash\{0\}$ *and* $\alpha + \beta < 0$*:*

(a) *For* $\beta \leq -|\alpha|$*, the system is exponentially stable independent of the value of* $\tau \in \mathbb{R}_{\geq 0}$*, i.e.,* $\bar{\tau} = \infty$.
(b) *For* $\alpha < -|\beta|$*, the system is exponentially stable if and only if* $\tau \in [0, \bar{\tau})$ *where*

$$\bar{\tau} = \frac{\arccos(-\frac{\beta}{\alpha})}{\sqrt{\alpha^2 - \beta^2}}. \tag{11}$$

The tightest estimate of the rate of convergence of (8) is characterized by the magnitude of the real part of the rightmost root of its characteristic equation $s = \frac{1}{\tau} W_0(\alpha\, \tau\, \mathrm{e}^{-\tau\beta}) + \beta$ (recall Lemma 2.1 and (2a)). That is (see [24, Corollary 1])

$$\rho_\tau = -\frac{1}{\tau}\, \mathrm{Re}(W_0(\alpha\, \tau\, \mathrm{e}^{-\tau\beta})) - \beta. \tag{12}$$

To compare the rate of convergence (12) to the delay free rate, we define the *delay gain function*

$$g(\gamma, \mathsf{x}) = \begin{cases} \frac{1}{\mathsf{x}}\, \mathrm{Re}(W_0(\mathsf{x}\, \mathrm{e}^{\gamma\, \mathsf{x}})), & \mathsf{x} \in \mathbb{R}\backslash\{0\}, \\ 1, & \mathsf{x} = 0, \end{cases} \tag{13}$$

where $\mathsf{x}, \gamma \in \mathbb{R}$. Then, we write (12) as

$$\rho_\tau = -(g(\gamma, \mathsf{x})\, \alpha + \beta) = -(g(\gamma, \mathsf{x}) - \gamma)\, \alpha \tag{14}$$

where $\mathsf{x} = \alpha\tau$ and $\gamma = -\frac{\beta}{\alpha}$. Note here that by invoking (1a), we obtain

$$\lim_{\tau \to 0} g(\gamma, \alpha\tau) = 1. \tag{15}$$

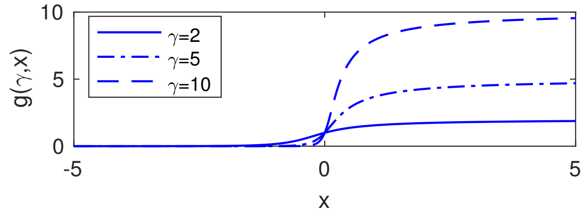
Therefore, as expected, $\lim_{\tau \to 0} \rho_\tau = \rho_0$, where

$$\rho_0 = -(\alpha + \beta) = -(1 - \gamma)\alpha. \tag{16}$$

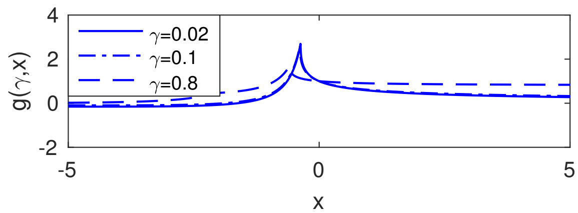
Knowing $\alpha + \beta < 0$, *the continuity stability property* theorem [29, Proposition 3.1] guarantees existence of the *admissible range* of delay $[0, \bar{\tau})$ for which the system (8) is exponentially stable. Next, we show that for some subset of the admissible delay range the delay gain $g(\gamma, \alpha\tau)$ for system (8) can be greater than 1, thus the delay results in an increase in the rate of convergence. The next lemma highlights some of the properties of the delay gain function $g(\gamma, \mathsf{x})$ (see Fig. 3).

Lemma 4.1 (Properties of $g(\gamma, \mathsf{x})$): *The following assertions hold for the delay gain function* (13) *with* $\gamma, \mathsf{x} \in \mathbb{R}$:

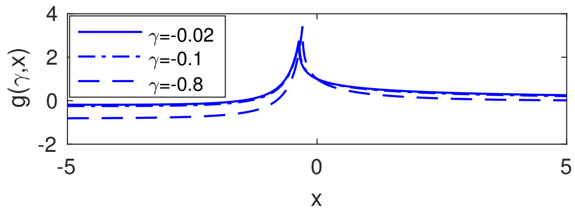
(a) For any $\gamma \in \mathbb{R}$ we have $\lim_{\mathsf{x} \to 0} g(\gamma, \mathsf{x}) = 1$.
(b) For any $\gamma > 1$ and $\mathsf{x} \in \mathbb{R}_{>0}$ we have $g(\gamma, \mathsf{x}) < \gamma$.
(c) For any $\gamma > 1$ and $\mathsf{x} \in \mathbb{R}_{>0}$, $g(\gamma, \mathsf{x})$ is a strictly increasing function of x.
(d) Let $\mathsf{x} \in (\bar{\mathsf{x}}, 0)$, where $\bar{\mathsf{x}} = \arccos(\gamma)/\sqrt{1 - \gamma^2}$. Then, for any $\gamma < 1$ (respectively $\gamma > 1$) we have $g(\gamma, \mathsf{x}) > \gamma$ (respectively $g(\gamma, \mathsf{x}) < \gamma$).
(e) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, $g(\gamma, \mathsf{x})$ is a strictly decreasing function of x for any $\mathsf{x} \in [\mathsf{x}^\star, 0) \subset (\bar{\mathsf{x}}, 0)$, and a strictly increasing function of x for any $\mathsf{x} < \mathsf{x}^\star$, where $\mathsf{x}^\star = \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$ when $\gamma \neq 0$ and $\mathsf{x}^\star = -\frac{1}{\mathrm{e}}$when $\gamma = 0$.
(f) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, the maximum value of $g(\gamma, \mathsf{x})$ occurs at $\mathsf{x}^\star = \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$ where $g(\gamma, \mathsf{x}^\star) =$



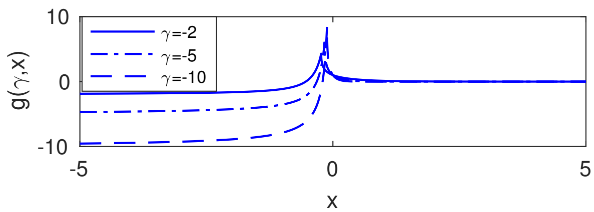
(a) $\gamma > 1$



(b) $0 < \gamma < 1$



(c) $-1 < \gamma < 0$



(d) $\gamma < -1$

Fig. 3: The delay gain function for different values of x, γ.

$\frac{-\gamma}{W_0(-\frac{\gamma}{e})}$ when $\gamma \neq 0$, and at $\mathsf{x}^\star = -\frac{1}{e}$ where $g(\gamma, \mathsf{x}^\star) = e$ when $\gamma = 0$.

(g) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, $g(\gamma, \mathsf{x}) > 1$ if and only if $\mathsf{x} \in (\tilde{\mathsf{x}}, 0)$ where $\tilde{\mathsf{x}}$ is the unique solution of $g(\gamma, \mathsf{x}) = 1$ in $(\bar{\mathsf{x}}, 0)$.

The prove this lemma we invoke various properties of the Lambert W function listed in Section II. The next theorem, whose proof relies on the results of Lemma 4.1 characterizes the effect of delay on the rate of convergence of system (8) in terms of different values of $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ satisfying $\alpha + \beta < 0$.

Theorem 4.2 (Effect of delay on the rate of convergence of delayed system (8)): *Consider system* (8) *with* $\alpha \in \mathbb{R}\backslash\{0\}$ *and* $\beta \in \mathbb{R}$ *such that* $\alpha + \beta < 0$, *whose rate of convergence* ρ_τ *is specified by* (14). *Consider also the delay gain function* (13) *with* $\gamma = -\frac{\beta}{\alpha}$ *and* $\mathsf{x} = \alpha\tau$. *Then,*

(a) *for* $\alpha > 0$ *and* $\beta < 0$ *the system* (8) *is exponentially stable for any* $\tau \in \mathbb{R}_{\geq 0}$. *Moreover, the rate of convergence decreases by increasing* $\tau \in \mathbb{R}_{\geq 0}$.

(b) *for* $\alpha < 0$ *and* $\beta \in \mathbb{R}$, $\rho_\tau > \rho_0$ *if and only if* $\tau \in [0, \tilde{\tau}) \subset [0, \bar{\tau})$ *where* $\tilde{\tau}$ *is the unique solution of* $g(\gamma, \alpha\tau) = 1$ *in* $(0, \bar{\tau})$ *and* $\bar{\tau}$ *is specified by*

$$\bar{\tau} = \arccos(-\beta/\alpha)/\sqrt{\alpha^2 - \beta^2}. \tag{17}$$

Moreover, ρ_τ *is monotonically increasing (resp. decreasing) with* τ *for any* $\tau \in [0, \tau^\star) \subset [0, \bar{\tau})$ *(resp.* $\tau \in (\tau^\star, \bar{\tau}) \subset [0, \bar{\tau})$*), where* $\tau^\star = -\frac{1}{\beta} W_0(\frac{\beta}{\alpha\, e})$ *when* $\beta \neq 0$ *and* $\tau^\star = -\frac{1}{\alpha\, e}$ *when* $\beta = 0$. *Finally, the maximum rate of convergence of* $\rho^\star_\tau = -(1 + \frac{1}{W_0(\frac{\beta}{\alpha\, e})})\beta$ *when* $\beta \neq 0$ *and* $\rho^\star_\tau = -\alpha\, e$ *when* $\beta = 0$ *is obtained at* $\tau = \tau^\star$.

To develop our main results in next section, we also invoke the following result.

Lemma 4.2: (maximum value of the trajectory of (8) [30, Theorem 2.10]) For the time delay system (8) and any $\tau \in (0, \bar{\tau}]$ with $\alpha, \beta \in \mathbb{R}_{<0}$ the following holds

$$|x(t)|_\infty = \max_{s \in [-\tau, 2\tau]} |x(s)|. \tag{18}$$

V. Main Result

In this section, we examine the convergence properties and the stability conditions of the modified average consensus algorithm (6) for all value of $\mathsf{k} \in \mathbb{R}/\{0\}$. Our focus is to show that for any $\mathsf{k} \in \mathbb{R}_{>0}$, there exists a range of delay for which the convergence rate of algorithm (6) improves. Also, we demonstrate that for any $\mathsf{k} \in (0, 1]$ higher convergence rate is achieved without increasing the control effort. We start our study by identifying the admissible range (values of delay for which the stability of algorithm (6) is preserved).

Lemma 5.1 (Admissible rage of delay for internal stability of algorithm (6)): *The following assertions hold for the modified average consensus algorithm* (6) *over an undirected connected graph:*

(a) *for* $\mathsf{k} \leq 0.5$, *the modified average consensus algorithm* (6) *is stable. Moreover, under the stated initial conditions,* x^i, $i \in \mathcal{V}$, *converges exponentially fast to* $\mathsf{x}^{avg}(0)$ *for any* $\tau \in \mathbb{R}_{\geq 0}$, *i.e.,* $\bar{\tau} = \infty$.

(b) *for* $\mathsf{k} > 0.5$, *the modified average consensus algorithm* (6) *is stable if and only if* $\tau \in [0, \bar{\tau})$, *where*

$$\bar{\tau} = \arccos(1 - 1/\mathsf{k})/(|\alpha|\lambda_N\sqrt{2\mathsf{k} - 1}). \tag{19}$$

Moreover, under the stated initial conditions, x^i, $i \in \mathcal{V}$, *converges exponentially fast to* $\mathsf{x}^{avg}(0)$.

Proof: [Sketch of the proof] Given (7) as the equivalent representation of algorithm (6), the proof can be obtained by invoking the statements (a) and (b) of Theorem 4.1. ■

Recall that in the absence of the delay, the rate of convergence of the average consensus algorithm (5) is $\rho_0 = \alpha\lambda_2$. In the admissible range of delay, using the modified average consensus algorithm (6), the rate of convergence of the algorithm is

$$\rho_\tau = \min\{\rho_{\tau,i}\}_{i=2}^N, \tag{20}$$

$$\rho_{\tau,i} = (\mathsf{k}g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau) + (1 - \mathsf{k}))\,\alpha\lambda_i.$$

The following result uses the properties of the delay gain function developed in Section IV to determine the range of delay for which $\rho_\tau > \alpha\lambda_2$ for a given k. We also identify the optimum value of the delay $\tau^\star$ for which ρ_τ has its maximum value, i.e., we identify the solution for

$$\tau^\star = \underset{\tau\in(0,\bar{\tau})}{\operatorname{argmax}}\, \rho_\tau = \underset{\tau\in(0,\bar{\tau})}{\operatorname{argmax}}\, \min\{\rho_{\tau,i}\}_{i=2}^{N}. \tag{21}$$

Here, to simplify the notation we wrote $\rho_{\tau,\mathsf{k}}$ as ρ_τ. In what follows, we let $\bar{\tau}_i$ be the critical delay value for z_i dynamics (7b), and

$$\tau_i^\star = \underset{\tau\in(0,\bar{\tau}_i)}{\operatorname{argmax}}\, \rho_{\tau,i},$$
$$\tilde{\tau}_i = \{\tau \in (0,\bar{\tau}_i) \,|\, g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_i\tau)) = 1\},$$

for $i \in \{2,\cdots,N\}$. The next theorem examines the effect of outdated feedback on the rate of convergence of modified consensus algorithm (6) for different values of $\mathsf{k} \in \mathbb{R}/\{0\}$.

Theorem 5.1 (Effect of outdated feedback on the rate of convergence of algorithm (6)): *The following assertions hold for the modified average consensus dynamics* (6) *over a connected graph whose rate of convergence is specified in* (20)*:*

(a) *For $\mathsf{k} < 0$ the rate of convergence of the consensus algorithm* (6) *decreases by increasing $\tau \in \mathbb{R}_{\geq 0}$.*
(b) *For $\mathsf{k} > 0$, $\rho_\tau > \rho_0$ if and only if $\tau \in [0,\hat{\tau}) \subset [0,\bar{\tau})$ where $\hat{\tau} = \min\{\hat{\tau}_i\}_{i=2}^N$ with $\hat{\tau}_i = \{\tau \in \mathbb{R}_{>0} | \rho_{\tau,i} = \rho_0\}$ and satisfies $\tilde{\tau}_N \leq \hat{\tau} \leq \min\{\tilde{\tau}_2, \bar{\tau}\}$. Moreover, the optimum time delay $\tau^\star$ corresponding to the maximum rate of convergence of the consensus algorithm* (6) *satisfies $\tau^\star \in [\tau_N^\star, \min\{\tau_2^\star, \hat{\tau}\}]$, where $\tau_N^\star = \frac{1}{\alpha(1-\mathsf{k})\lambda_N} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})$ and $\tau_2^\star = \frac{1}{\alpha(1-\mathsf{k})\lambda_2} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})$, and is given by $\tau^\star = \{\tau \in [\tau_N^\star, \min\{\tau_2^\star, \hat{\tau}\}] \,|\, \rho_{\tau,2} = \min\{\rho_{\tau,i}\}_{i=3}^N\}$.*

Proof: [Sketch of the proof] Recall (20), the proof of part (a) follows from statement (a) of Theorem 4.2 . To prove statement (b) note that for $\mathsf{k} > 0$, because of the statement (b) of Theorem 4.2 for each z_i, $i \in \{2,\cdots,N\}$, dynamics ($-\alpha\mathsf{k}\lambda_i < 0$) we have the guarantees that for $\tau \in (0,\tilde{\tau}_i)$

$$\rho_{\tau,i} = (\mathsf{k}g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau) + (1-\mathsf{k}))\,\alpha\lambda_i > \rho_{0,i} \geq \rho_0.$$

Since $g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau)$ is a decreasing function of τ for any $\tau \in (\tilde{\tau}_i, \bar{\tau}_i) \subset (\tau_i^\star, \bar{\tau}_i)$ (Recall Lemma 4.1), it follows that for any $\tau \in [0,\hat{\tau}_j)$ we have $\rho_{\tau,j} > \rho_0$ and for any $\tau \in [\hat{\tau}_j, \bar{\tau})$ we have $\rho_{\tau,j} < \rho_0$. Also, note that the maximum value of ρ_τ is attained at $\tau = \tau^\star$ at which

$$\min\{\rho_{\tau,i}\}_{i=\mathsf{j}}^N = \min\{\rho_{\tau,i}\}_{i=2}^{\mathsf{j}-1}. \tag{22}$$

Since $\lambda_2\tau^\star \leq \cdots \leq \lambda_{\mathsf{j}-1}\tau^\star$ and $\mathrm{d}g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_i\tau^\star)/\mathrm{d}\tau > 0$ for $i \in \{2,\cdots,\mathsf{j}-1\}$, we have $g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_{\mathsf{j}-1}\tau^\star) \geq g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_{\mathsf{j}-2}\tau^\star) \geq \cdots \geq g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_2\tau^\star)$. As a result, it follows from (20) that at $\tau = \tau^\star$ we have $\min\{\rho_{\tau,i}\}_{i=2}^{\mathsf{j}-1} = \rho_{\tau,2}$. ■

Theorem 5.1 implies that for any $\mathsf{k} > 0$ there exists a range in $(0,\bar{\tau}]$ in which faster response can be achieved for the modified average consensus algorithm (6) relative to the original one (5). Next, our goal is to specify values of $\mathsf{k} \in \mathbb{R}_{>0}$ for which maximum control effort of the agents do not exceed the one for the delay free algorithm (5). The next theorem demonstrates that for any $\mathsf{k} \in (0,1]$ using the outdated feedback does not increase the maximum control effort while for $k > 1$ the maximum control effort is greater than the one for the original algorithm (5).

Theorem 5.2 (The maximum control effort for the algorithm (6)): *Let $\mathbf{u}_{\tau,\mathsf{k}}(t) = \dot{\mathbf{x}}(t)$ for the modified distributed algorithm* (6) *with $\mathsf{k} \in \mathbb{R}_{>0}$. Then, for any $\tau \in [0,\bar{\tau}]$, where admissible delay bound τ is given in Lemma 5.1, the following assertions hold for $t \in \mathbb{R}_{\geq 0}$:*

(a) *For $\mathsf{k} \in (0,1]$ we have $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \leq |\mathbf{u}_{0,0}(t)|_\infty$.*
(b) *For $\mathsf{k} > 1$ we have $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \geq \mathrm{e}^{(k-1)\alpha\lambda_2\tau}\,|\mathbf{u}_{0,0}(t)|_\infty$.*

Proof: [Sketch of the proof] Using the transformation matrix (4), for the maximum control effort of algorithm (6) we have

$$\begin{aligned}|\mathbf{u}_{\tau,k}(t)|_\infty &= |-\alpha\,(1-\mathsf{k})\,\mathbf{\Lambda}\,\mathbf{z}(t) - \alpha\,\mathsf{k}\,\mathbf{\Lambda}\,\mathbf{z}(t-\tau)|_\infty \\ &= \alpha\max\{|(1-\mathsf{k})\,\lambda_i\,z_i(t) + \mathsf{k}\,\lambda_i\,z_i(t-\tau)|_\infty\}_{i=2}^N.\end{aligned} \tag{23}$$

Note that $z_1(t) = 0$. Also, recalling (7b), for $\tau = 0$ and any $i \in \{2,\cdots,N\}$ we have $z_i(t) = \mathrm{e}^{-\lambda_i t}\,z_i(0)$, which gives $|\mathbf{u}_{0,0}(t)|_\infty = |\mathbf{u}_{0,0}(0)|_\infty = \alpha\max\{|\lambda_i z_i(0)|\}_{i=2}^N$. The rest of the proof of part (a) can be proceed by using Lemma 4.2 to show that $|z_i(t)|_\infty = |z_i(0)|$ which gives $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \leq \alpha\{|\lambda_i z_i(0)|\}_{i=2}^N$. Recalling (23) for $\mathsf{k} > 1$ we have $|\mathbf{u}_{\tau,\mathsf{k}}(2\tau)| \geq \alpha(2\mathsf{k}-1)\,\mathrm{e}^{\alpha(\mathsf{k}-1)\lambda_i\tau}\max\{|\lambda_i z_i(0)|\}_{i=2}^N = (2\mathsf{k}-1)\,\mathrm{e}^{\alpha(\mathsf{k}-1)\lambda_2\tau}\,|\mathbf{u}_{0,0}|_\infty$. Knowing $2\mathsf{k}-1 \geq 1$ and $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \geq |\mathbf{u}_{\tau,\mathsf{k}}(2\tau)|_\infty$ concludes statement (b). ■

VI. Numerical Example

Consider the modified average consensus algorithm (6) over the graph depicted in Fig. 2. The rate of convergence ρ_τ of this execution versus τ for feedback gain $\mathsf{k} \in \{0, 0.5, 1, 1.5\}$ is shown in Fig. 4. As seen, for none-zero values of k the rate of convergence first increases, and after reaching a maximum value then it decreases with $\tau \in \mathbb{R}_{>0}$, as mentioned in Theorem 5.1 part (b). Moreover, as predicted in Theorem 5.1, for any $\mathsf{k} \in \{0.5, 1, 1.5\}$, there exists a range of τ for which $\rho_\tau > \rho_0$. In addition, the maximum achievable rate of convergence is larger for larger values of k (for $\mathsf{k} = 1.5$ we have $\tilde{\tau} = 0.14$ and the maximum rate of convergence $\rho_\tau^\star \approx 2\rho_0$ at $\tau^\star = 0.11$). Figure 5 shows the maximum control effort of algorithm (6) over time for $\tau = 0.1$ and $\mathsf{k} \in \{0.5, 1, 1.5\}$ and the delay free case of $\mathsf{k} = 0$. For $\mathsf{k} = 1.5$ the maximum control effort exceeds the value for the delay free case of $\mathsf{k} = 0$. But, for $\mathsf{k} = 1$ and $\mathsf{k} = 0.5$ the maximum control effort is equal or less than the case $\mathsf{k} = 0$. These observations are in accordance with the results in Theorem 5.2.

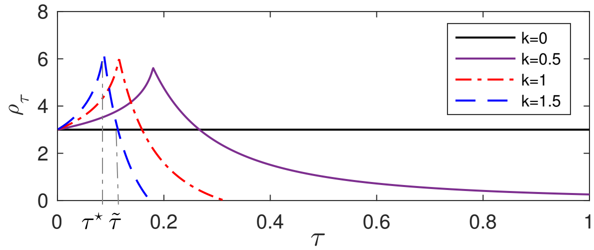


Fig. 4: The rate of convergence ρ_τ of the modified average consensus algorithm (6) over the graph in Fig. 2 for different values of feedback gain $\mathsf{k}=0$, $\mathsf{k}=0.5$, $\mathsf{k}=1$ and $\mathsf{k}=1.5$.

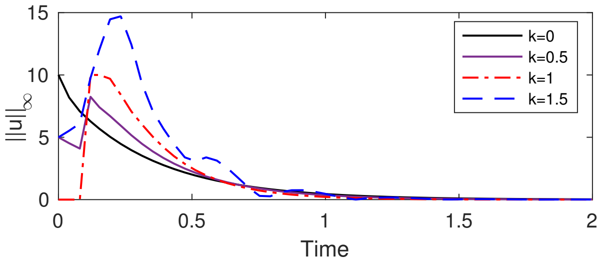


Fig. 5: The maximum control effort executed by the algorithm (6) over the graph in Fig. 2 for $\tau=0.1$ and different values of feedback gain $\mathsf{k}=0$, $\mathsf{k}=0.5$, $\mathsf{k}=1$ and $\mathsf{k}=1.5$.

VII. Conclusion

We analyzed the effect of a weighted outdated feedback in increasing the rate of convergence of the Laplacian average consensus algorithm. Our study produced a set of closed-form expressions to specify the admissible delay range, the delay range for which the system experiences increase in its rate of convergence and the optimum value of delay corresponding to the maximum rate of convergence. We also studies for what ranges of the outdated disagreement feedback gain the rate of convergence of the Laplacian average consensus algorithm can increase without increasing the control effort. To develop our results we used the Lambert W function. Our future work includes investigating the role of delayed feedback in increasing the rate of convergence of other distributed algorithms for networked systems.

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